

WOJCIECH BUSZKOWSKI  
EWA PALKA

# Infinitary Action Logic: Complexity, Models and Grammars

**Abstract.** Action logic of Pratt [21] can be presented as Full Lambek Calculus FL [14, 17] enriched with Kleene star  $*$ ; it is equivalent to the equational theory of residuated Kleene algebras (lattices). Some results on axiom systems, complexity and models of this logic were obtained in [4, 3, 18]. Here we prove a stronger form of  $*$ -elimination for the logic of  $*$ -continuous action lattices and the  $\Pi_1^0$ -completeness of the equational theories of action lattices of subsets of a finite monoid and action lattices of binary relations on a finite universe. We also discuss possible applications in linguistics.

*Keywords:* Kleene algebra, action algebra, relation algebra, categorial grammar.

## 1. Introduction

Kleene algebras are idempotent semirings with Kleene star  $*$  (Kozen [10, 11]). Action algebras are residuated Kleene algebras. They were introduced by Pratt [21] to provide an equational axiomatization of the equations valid in Kleene algebras; by the Kozen completeness theorem [11], these equations are precisely the equations true for regular expressions. Action algebras are algebraic models of Full Lambek Calculus (FL) with  $*$  but without  $\wedge$ . With  $\wedge$ , they are called action lattices. Concrete action algebras and lattices appear in mathematical linguistics (algebras of languages) and logics of programs (algebras of relations).

Action Logic (ACT) is a propositional logic, corresponding to action algebras (equivalent to the equational theory of action algebras). It is not known whether ACT admits a cut-free sequent axiomatization and is decidable [9].

A Kleene algebra is  $*$ -continuous, if  $xa^*y = \bigvee\{xa^ny : n \in \omega\}$ , for all elements  $x, a, y$ . Algebras of languages and algebras of relations are  $*$ -continuous. From the Kozen completeness theorem it follows that the equational theory of all Kleene algebras is decidable and equals that of  $*$ -continuous Kleene algebras; they amount to the equational theory of relational Kleene algebras [13].

In [4, 3], it has been shown that the situation is different for the case of action algebras (lattices). The equational theory of  $*$ -continuous action

---

Presented by **Jacek Malinowski**; *Received* September 9, 2007

algebras (lattices) is  $\Pi_1^0$ -complete, whence it strictly contains the equational theory of all action algebras (lattices) which is  $\Sigma_1^0$ . The former possesses Finite Model Property (FMP), but the latter does not (otherwise, it would be equal to the former, by the  $*$ -continuity of finite action algebras). The equational theory of Kleene algebras possesses FMP [19]. The equational theory of action algebras of relations is  $\Pi_1^0$ -hard. The equational theory of algebras of regular languages is  $\Pi_1^0$ -complete. The equational theory of algebras of languages is not in  $\Sigma_1^0 \cup \Pi_1^0$ .

The logic of  $*$ -continuous action lattices ( $\text{ACT}\omega$ ) is an infinitary logic: an extension of FL by some rules for  $*$ , and one of them is an  $\omega$ -rule [4]. The cut-elimination theorem and a theorem on  $*$ -elimination for  $\text{ACT}\omega$  are proved in [18]. As a consequence of the latter theorem,  $\text{ACT}\omega$  is  $\Pi_1^0$ . It is  $\Pi_1^0$ -hard, since the total language problem for context-free grammars is reducible to it, using categorial grammars and cut-elimination for FL [4]. The same reduction yields the  $\Pi_1^0$ -hardness of other theories of action algebras, mentioned above.

The present paper continues this research. We obtain some new results and discuss certain applications of action logic.

In section 2, we define basic algebraic notions and present the infinitary system  $\text{ACT}\omega$ . The  $*$ -elimination theorem from [18] is strengthened: we eliminate all (not only negative) occurrences of  $*$ . We also sketch a different proof of the elimination of negative occurrences of  $*$ . In section 3, we use the reduction, applied in [4] in the proof of  $\Pi_1^0$ -hardness of  $\text{ACT}\omega$ , to prove the  $\Pi_1^0$ -completeness of the equational theories of powerset action lattices over finite monoids and finite relational (square) action lattices. We briefly announce other results of that kind. In section 4, we discuss certain problems, connected with applications of action logic in categorial grammars.

Theorems 4 and 5 are common results of both authors; they have been included in PhD Thesis of the second author [19] with slightly different proofs. The new proof of theorem 1 and the remaining results of this paper are due to the first author.

## 2. Action logic and $*$ -elimination

A *Kleene algebra* is an algebra  $\mathcal{M} = (M, \vee, \cdot, *, 0, 1)$  such that  $(M, \vee)$  is a (join) semilattice, 0 is its lower bound,  $(M, \cdot, 1)$  is a monoid, product  $\cdot$  distributes over  $\vee$ ,  $0a = 0 = a0$ , and  $*$  is a unary operation, satisfying:

$$1 \vee aa^* \leq a^*; 1 \vee a^*a \leq a^*, \quad (1)$$

$$\text{if } ab \leq b \text{ then } a^*b \leq b; \text{ if } ba \leq b \text{ then } ba^* \leq b, \quad (2)$$

for all  $a, b \in M$ . One defines:  $a \leq b$  iff  $a \vee b = b$ . Many authors write  $+$  for  $\vee$ , but  $\vee$  is more appropriate for our purposes.

This notion is due to Kozen [10, 11] as an algebraic counterpart of the (informal) algebra of regular expressions. Regular expressions can be defined as terms of the first-order language of Kleene algebras with variables being replaced by symbols from an alphabet  $\Sigma$ .  $\Sigma^*$  denotes the set of all finite strings on  $\Sigma$  and  $\epsilon$  denotes the empty string. Hence  $\Sigma^*$  with concatenation and  $\epsilon$  is the free monoid generated by  $\Sigma$ . Subsets of  $\Sigma^*$  are called *languages* on  $\Sigma$ . Each regular expression  $\alpha$  on  $\Sigma$  determines (or: denotes) a language  $L(\alpha) \subseteq \Sigma^*$ , defined by:  $L(a) = \{a\}$ , for  $a \in \Sigma$ ,  $L(0) = \emptyset$ ,  $L(1) = \{\epsilon\}$ ,  $L(\alpha \vee \beta) = L(\alpha) \cup L(\beta)$ ,  $L(\alpha \cdot \beta) = L(\alpha)L(\beta)$ ,  $L(\alpha^*) = (L(\alpha))^*$ . So,  $L(\cdot)$  is the assignment of regular expressions in the Kleene algebra  $P(\Sigma^*)$ , determined by  $L(a) = \{a\}$ , for  $a \in \Sigma$  (see section 3 for the definition of operations on the powerset of a monoid). Languages  $L(\alpha)$ , for regular expressions  $\alpha$ , are called *regular languages*. The Kozen completeness theorem [11] states that  $L(\alpha) = L(\beta)$  iff  $\alpha = \beta$  is valid in Kleene algebras.

If  $\mathcal{K}$  is a class of algebras, then  $\text{Eq}(\mathcal{K})$  denotes the equational theory of  $\mathcal{K}$ , i.e. the set of all equations valid in  $\mathcal{K}$ . KA and KA\* denote the class of all Kleene algebras and the class of \*-continuous Kleene algebras, respectively, and similarly for other classes. REL denotes the class of Kleene algebras  $P(U^2)$ , consisting of binary relations on a set  $U$ . It follows from the Kozen theorem that  $\text{Eq}(\text{KA}) = \text{Eq}(\text{KA}^*) = \text{Eq}(\text{REL})$  ( $L(\alpha) = L(\beta)$  iff  $\alpha = \beta$  is valid in REL [13]).

An *action algebra* is a Kleene algebra  $\mathcal{M}$  with two binary operations  $\rightarrow$  (right residuation) and  $\leftarrow$  (left residuation), satisfying:

$$ab \leq c \text{ iff } b \leq a \rightarrow c \text{ iff } a \leq c \leftarrow b, \quad (3)$$

for all  $a, b, c \in M$ . This notion is due to Pratt [21]. The set  $P(\Sigma^*)$ , of languages on  $\Sigma$ , is an action algebra with  $\vee, \cdot, *, 0, 1$  defined as usual (see section 3) and  $L_1 \rightarrow L_2 = \{x \in \Sigma^* : L_1\{x\} \subseteq L_2\}$ ,  $L_2 \leftarrow L_1 = \{x \in \Sigma^* : \{x\}L_1 \subseteq L_2\}$ .  $P(U^2)$  is an action algebra with  $R \rightarrow S = \{(x, y) \in U^2 : R \circ \{(x, y)\} \subseteq S\}$ ,  $S \leftarrow R = \{(x, y) \in U^2 : \{(x, y)\} \circ R \subseteq S\}$ . ACTA denotes the class of action algebras. It follows from the Kozen theorem that  $\text{Eq}(\text{KA})$  and  $\text{Eq}(\text{ACTA})$  contain the same residuation-free equations. Pratt [21] shows that ACTA admits a finite, equational axiomatization; KA is not a variety.

An *action lattice* is an action algebra  $\mathcal{M}$  with meet  $\wedge$  such that  $(M, \vee, \wedge)$  is a lattice. Kleene lattices are defined in a similar way. ACTL and KL denote the classes of action lattices and Kleene lattices, respectively. It is not known whether  $\text{Eq}(\text{ACTA})$ ,  $\text{Eq}(\text{ACTL})$  are decidable [9].

In action algebras (not in Kleene algebras, in general), product distributes over infinite joins. Consequently, an action algebra is  $*$ -continuous iff  $a^* = \bigvee \{a^n : n \in \omega\}$ , for all  $a$ . Natural action algebras  $P(\Sigma^*)$  and  $P(U^2)$  are  $*$ -continuous. Every complete action lattice is  $*$ -continuous, and  $*$ -continuous action lattices are precisely the sublattices of complete action lattices [4].

Pratt's axiomatizations of  $\text{Eq}(\text{ACTA})$  and  $\text{Eq}(\text{ACTL})$  show that these theories are  $\Sigma_1^0$ . It would be interesting to find a more effective axiomatization. Jipsen [9] proposes a Gentzen-style sequent system which extends Full Lambek Calculus FL, but it does not admit cut elimination [4], whence does not yield any proof-search algorithm. With cut,  $\text{Eq}(\text{ACTL})$  can be axiomatized quite directly.

First, we recall the standard axiomatization of FL [17]. Formulas are  $*$ -free terms of the first-order language of ACTL. Sequents are of the form  $\Gamma \Rightarrow \alpha$ , where  $\Gamma$  is a finite sequence of formulas, and  $\alpha$  is a formula (so, this is an intuitionistic sequent system). FL admits the following axioms and rules:

$$\begin{aligned}
& (\text{Id}) \alpha \Rightarrow \alpha, \quad (0\text{L}) \Gamma, 0, \Delta \Rightarrow \alpha, \quad (1\text{R}) \Rightarrow 1, \\
& (\vee\text{L}) \frac{\Gamma, \alpha, \Delta \Rightarrow \gamma; \Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \gamma}, \quad (\vee\text{R}) \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}, \\
& (\wedge\text{L}) \frac{\Gamma, \alpha_i, \Delta \Rightarrow \beta}{\Gamma, \alpha_1 \wedge \alpha_2, \Delta \Rightarrow \beta}, \quad (\wedge\text{R}) \frac{\Gamma \Rightarrow \alpha; \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}, \\
& (\cdot\text{L}) \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma}, \quad (\cdot\text{R}) \frac{\Gamma \Rightarrow \alpha; \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}, \\
& (\rightarrow\text{L}) \frac{\Gamma, \beta, \Delta \Rightarrow \gamma; \Phi \Rightarrow \alpha}{\Gamma, \Phi, \alpha \rightarrow \beta, \Delta \Rightarrow \gamma}, \quad (\rightarrow\text{R}) \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}, \\
& (\leftarrow\text{L}) \frac{\Gamma, \beta, \Delta \Rightarrow \gamma; \Phi \Rightarrow \alpha}{\Gamma, \beta \leftarrow \alpha, \Phi, \Delta \Rightarrow \gamma}, \quad (\leftarrow\text{R}) \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta \leftarrow \alpha}, \\
& (1\text{L}) \frac{\Gamma, \Delta \Rightarrow \alpha}{\Gamma, 1, \Delta \Rightarrow \alpha}.
\end{aligned}$$

We write  $\text{FL} \vdash \Gamma \Rightarrow \alpha$ , if  $\Gamma \Rightarrow \alpha$  is provable in FL, and similarly for other systems. The cut rule:

$$(\text{CUT}) \frac{\Gamma, \alpha, \Delta \Rightarrow \beta; \Phi \Rightarrow \alpha}{\Gamma, \Phi, \Delta \Rightarrow \beta}$$

is admissible in FL [17], this means: if the premises are provable, then the conclusion is provable. The first proof of cut elimination for systems

of that kind was given by Lambek [14] for the  $(\cdot, \rightarrow, \leftarrow)$ -fragment of FL (the Lambek calculus  $L^*$ ); actually, Lambek considered a weaker system, admitting no sequents with the empty antecedent. FL is complete with respect to residuated lattices [17], i.e. algebras  $(M, \vee, \wedge, \cdot, \rightarrow, \leftarrow, 0, 1)$  such that  $(M, \vee, \wedge, 0)$  is a lattice with lower bound 0,  $(M, \cdot, 1)$  is a monoid, and  $\cdot, \rightarrow, \leftarrow$  satisfy (3). It means that  $FL \vdash \alpha_1, \dots, \alpha_n \Rightarrow \alpha$  iff  $\alpha_1 \cdots \alpha_n \leq \alpha$  is valid in residuated lattices.

The logic ACT can be axiomatized as FL, extended for formulas with  $*$ , with (CUT) and the following axioms and rules for  $*$ : (\*R1)  $\Rightarrow \alpha^*$ , (\*A1)  $\alpha, \alpha^* \Rightarrow \alpha^*$ , (\*A2)  $\alpha^*, \alpha \Rightarrow \alpha^*$ , (\*L1) from  $\alpha, \beta \Rightarrow \beta$  infer  $\alpha^*, \beta \Rightarrow \beta$ , (\*L2) from  $\beta, \alpha \Rightarrow \beta$  infer  $\beta, \alpha^* \Rightarrow \beta$ ; they correspond to (1), (2). ACT is complete with respect to action lattices, and its  $\wedge$ -free fragment is complete with respect to action algebras. ACT may be treated as an axiomatization of Eq(CTL). Each order formula  $\alpha \leq \beta$  represents an equation; each equation  $\alpha = \beta$  can be represented by the conjunction of  $\alpha \leq \beta$  and  $\beta \leq \alpha$ . Then, ACT and Eq(CTL) have the same complexity. This also holds for other systems, considered later on.

The logic  $ACT\omega$  can be axiomatized as FL (language extended as above) with (\*R1) and the following rules:

$$(*R2) \frac{\Gamma_1 \Rightarrow \alpha; \dots; \Gamma_n \Rightarrow \alpha}{\Gamma_1, \dots, \Gamma_n \Rightarrow \alpha^*},$$

$$(*L) \frac{(\Gamma, \alpha^n, \Delta \Rightarrow \beta)_{n \in \omega}}{\Gamma, \alpha^*, \Delta \Rightarrow \beta}.$$

In (\*R2) we assume  $n \geq 1$  and  $\Gamma_i \neq \epsilon$ , for  $i = 1, \dots, n$ ; the scheme represents an infinite family of finitary rules. (\*L) is an infinitary rule (an  $\omega$ -rule). Axioms (Id) can be restricted to the form  $p \Rightarrow p$ , for all variables  $p$ .

The set of sequents provable in  $ACT\omega$  can be defined as the limit of a chain  $S_\xi$ , for  $\xi < \omega_1$ , such that  $S_0 = \emptyset$ ,  $S_{\xi+1}$  is the closure of  $S_\xi$  under a single application of each rule (axioms are treated as rules with no premises), and  $S_\lambda$  is the limit of  $S_\xi$ ,  $\xi < \lambda$ , for limit ordinals  $\lambda$ .

(CUT) is admissible in  $ACT\omega$  [18]. Consequently, the subformula property holds: every provable sequent is provable in the fragment of  $ACT\omega$ , whose sequents only consist of subformulas of formulas appearing in this sequent. FL is a conservative fragment of  $ACT\omega$ . The  $(\rightarrow, \leftarrow)$ -free fragment of  $ACT\omega$  is a cut-free sequent system for  $Eq(KA^*) = Eq(KA)$ . Using (CUT), one easily proves that  $ACT\omega$  is complete with respect to  $*$ -continuous action lattices [18].

Positive and negative occurrences of subformulas in a formula are defined as usual; in  $\gamma \equiv \alpha \rightarrow \beta$  or  $\gamma = \beta \leftarrow \alpha$ , each positive (resp. negative) subformula of  $\beta$  is positive (resp. negative) in  $\gamma$ , and conversely for subformulas of  $\alpha$  (other connectives preserve polarity). For a sequent  $\Gamma \Rightarrow \alpha$ , each positive (resp. negative) subformula of a formula from  $\Gamma$  is negative (resp. positive) in the sequent, and conversely for subformulas of  $\alpha$  (this reflects the fact that  $\gamma \Rightarrow \alpha$  is provable iff  $\Rightarrow \gamma \rightarrow \alpha$  is provable). An occurrence of a connective is positive (resp. negative) in a formula (sequent), if it is the main connective of a subformula which is positive (resp. negative) in the formula (sequent).

For any  $n \in \omega$ , we define mappings  $P_n$  (resp.  $N_n$ ) which transform any formula  $\alpha$  into a formula with no positive (resp. negative) occurrence of  $*$ . Let  $\alpha^{\leq n} \equiv \alpha^0 \vee \alpha^1 \vee \dots \vee \alpha^n$ , where  $\alpha^0 \equiv 1$ , and for  $n \geq 1$ ,  $\alpha^n$  is the product of  $n$  copies of  $\alpha$ .  $P_n(\alpha) = N_n(\alpha) = \alpha$ , if  $\alpha$  is a variable or a constant. Below  $\circ$  stands for any of the connectives  $\cdot, \vee, \wedge$ .

$$P_n(\alpha \circ \beta) = P_n(\alpha) \circ P_n(\beta), \quad N_n(\alpha \circ \beta) = N_n(\alpha) \circ N_n(\beta), \quad (4)$$

$$P_n(\alpha \rightarrow \beta) = N_n(\alpha) \rightarrow P_n(\beta), \quad N_n(\alpha \rightarrow \beta) = P_n(\alpha) \rightarrow N_n(\beta), \quad (5)$$

$$P_n(\beta \leftarrow \alpha) = P_n(\beta) \leftarrow N_n(\alpha), \quad N_n(\beta \leftarrow \alpha) = N_n(\beta) \leftarrow P_n(\alpha), \quad (6)$$

$$P_n(\alpha^*) = (P_n(\alpha))^{\leq n}, \quad N_n(\alpha^*) = (N_n(\alpha))^*. \quad (7)$$

For  $\Gamma = (\alpha_1, \dots, \alpha_k)$ , we set  $P_n(\Gamma) = (P_n(\alpha_1), \dots, P_n(\alpha_k))$ , and similarly for  $N_n(\Gamma)$ . We define  $P_n(\Gamma \Rightarrow \alpha) = N_n(\Gamma) \Rightarrow P_n(\alpha)$ ,  $N_n(\Gamma \Rightarrow \alpha) = P_n(\Gamma) \Rightarrow N_n(\alpha)$ .  $P_n(\Gamma \Rightarrow \alpha)$  (resp.  $N_n(\Gamma \Rightarrow \alpha)$ ) contains no positive (resp. negative) occurrence of  $*$ .

We have  $\text{ACT}\omega \vdash \alpha^n \Rightarrow \alpha^*$ , by (Id), (\*R1), (\*R2). By ( $\vee$ L),  $\text{ACT}\omega \vdash \alpha^{\leq m} \Rightarrow \alpha^{\leq n}$ , if  $m \leq n$ . Using these facts and (derivable) monotonicity rules of  $\text{ACT}\omega$ , e.g.  $\alpha \Rightarrow \beta$  and  $\gamma \Rightarrow \delta$  yield  $\delta \rightarrow \alpha \Rightarrow \gamma \rightarrow \beta$ , by induction on  $\alpha$ , one proves in  $\text{ACT}\omega$ :

$$P_n(\alpha) \Rightarrow \alpha, \quad \alpha \Rightarrow N_n(\alpha), \quad (8)$$

$$P_m(\alpha) \Rightarrow P_n(\alpha), \quad N_n(\alpha) \Rightarrow N_m(\alpha), \quad \text{for } m \leq n. \quad (9)$$

Using (CUT), one obtains:

- LEMMA 1. (L1) If  $\text{ACT}\omega \vdash \Gamma \Rightarrow \alpha$  then, for all  $n \in \omega$ ,  $\text{ACT}\omega \vdash N_n(\Gamma \Rightarrow \alpha)$ .  
(L2) For any  $n \in \omega$ , if  $\text{ACT}\omega \vdash P_n(\Gamma \Rightarrow \alpha)$  then  $\text{ACT}\omega \vdash \Gamma \Rightarrow \alpha$ .  
(L3) If  $m \leq n$  and  $\text{ACT}\omega \vdash P_m(\Gamma \Rightarrow \alpha)$ , then  $\text{ACT}\omega \vdash P_n(\Gamma \Rightarrow \alpha)$ .  
(L4) If  $m \leq n$  and  $\text{ACT}\omega \vdash N_n(\Gamma \Rightarrow \alpha)$ , then  $\text{ACT}\omega \vdash N_m(\Gamma \Rightarrow \alpha)$ .

Palka [18] proves the following theorem (below we give a different, shorter proof):

**THEOREM 1.**  $ACT\omega \vdash \Gamma \Rightarrow \alpha$  iff, for all  $n \in \omega$ ,  $ACT\omega \vdash N_n(\Gamma \Rightarrow \alpha)$ .

The ‘only if’ part follows from (L1). The ‘if’ part is proved by transfinite induction on a special complexity count of sequents:  $c(\Gamma \Rightarrow \alpha)$  equals  $(c_0, \dots, c_r)$ , where  $r$  is the maximal complexity of formulas appearing in  $\Gamma \Rightarrow \alpha$ , and  $c_i$  is the number of occurrences of formulas of complexity  $i$  in this sequent (the complexity of a formula is the total number of occurrences of connectives and constants in it). Finite sequences of integers are well-ordered in type  $\omega^\omega$  in the following way: shorter sequences are less than longer sequences; if  $c, d$  are different sequences of equal length, then  $c < d$  if  $c_i < d_i$ , for the greatest  $i$  such that  $c_i \neq d_i$ . It is easy to see that, for any rule of  $ACT\omega$ , the complexity of the conclusion is greater than the complexity of any premise.

Using this complexity count, one can show that the set of provable sequents of  $ACT\omega$  is the limit of  $S_\xi$ , for  $\xi < \omega^\omega$ . We sketch a different proof of the ‘if’ part of Theorem 1.

**PROOF.** Assume  $ACT\omega \vdash N_n(\Gamma \Rightarrow \alpha)$ , for all  $n \in \omega$ . We also assume that the thesis holds for all sequents of complexity less than  $c(\Gamma \Rightarrow \alpha)$ . We consider two cases.

**CASE 1.**  $\Gamma = \Gamma_1, \beta^*, \Gamma_2$ . Then,  $P_n(\Gamma_1), (P_n(\beta))^{\leq n}, P_n(\Gamma_2) \Rightarrow N_n(\alpha)$  is provable in  $ACT\omega$ , for all  $n \in \omega$ . Clearly,  $P_n(\Gamma_1), (P_n(\beta))^m, P_n(\Gamma_2) \Rightarrow N_n(\alpha)$  is provable, for all  $n \in \omega$  and all  $m \leq n$ .  $(P_n(\beta))^m = P_n(\beta^m)$ . Then  $N_n(\Gamma_1, \beta^m, \Gamma_2 \Rightarrow \alpha)$  is provable, for all  $n \geq m$ , whence it is provable for all  $n \in \omega$ , by (L4). Since  $c(\Gamma_1, \beta^m, \Gamma_2 \Rightarrow \alpha)$  is less than  $c(\Gamma \Rightarrow \alpha)$ , then the former sequent is provable, by the induction hypothesis. This holds for all  $m \in \omega$ , which yields the provability of  $\Gamma \Rightarrow \alpha$ , by (\*L).

**CASE 2.** No formula occurring in the sequence  $\Gamma$  is of the form  $\beta^*$ , for any  $\beta$ . We consider two subcases.

(2.1) For some  $n \in \omega$ ,  $N_n(\Gamma \Rightarrow \alpha)$  is an axiom of  $ACT\omega$  (we have restricted (Id) to variables). It is easy to see that  $\Gamma \Rightarrow \alpha$  must be an axiom of the same kind, whence  $\Gamma \Rightarrow \alpha$  is provable.

(2.2) For every  $n \in \omega$ ,  $N_n(\Gamma \Rightarrow \alpha)$  is a conclusion of some rule, and the premises are provable. Since  $N_n(\Gamma \Rightarrow \alpha)$  does not contain negative occurrences of  $*$ , this rule cannot be (\*L). So, it must be an instance of a finitary rule of the system. Notice that the length of the sequence  $P_n(\Gamma)$ , i.e. the number of formulas appearing in the sequence, equals that of  $\Gamma$ ; let us denote it by  $k$ . By  $p(n)$  we denote the position of the ‘active formula’,

i.e. the formula introduced by the rule, in  $N_n(\Gamma \Rightarrow \alpha)$ . Then,  $1 \leq p(n) \leq k$ , if the rule is an L-rule, and  $p(n) = k + 1$ , if the rule is an R-rule. There exists  $i \in \{1, \dots, k + 1\}$  such that  $p(n) = i$ , for infinitely many  $n$ . Fix such an  $i$ . Let  $\beta$  denote the formula which occurs on the  $i$ -th position in  $\Gamma \Rightarrow \alpha$  ( $\beta = \alpha$  if  $i = k + 1$ ). If  $i \leq k$  (resp.  $i = k + 1$ ), then the main connective in  $\beta$  equals the main connective in  $P_n(\beta)$  (resp.  $N_n(\beta)$ ), for all  $n$  (we use the assumption of Case 2). Then, for all  $n$  such that  $p(n) = i$ , the rule introducing  $N_n(\Gamma \Rightarrow \alpha)$  must introduce the same connective.

By the form of a rule we mean the sequence of lengths of antecedents of premises of the rule. For instance, for (\*R2), it is the sequence  $(l_1, \dots, l_n)$ , where  $l_i$  is the length of  $\Gamma_i$ , and for ( $\cdot$ R), it is the sequence  $(l_1, l_2)$ , where  $l_1$  is the length of  $\Gamma$  and  $l_2$  is the length of  $\Delta$ . For infinitely many  $n$  such that  $p(n) = i$ , the rule introducing  $N_n(\Gamma \Rightarrow \alpha)$  must be of the same form. We fix such a form. By scrutinizing all finitary rules, one easily sees that the premises of the rule (having the fixed form) are some sequents  $N_n(\Gamma_1 \Rightarrow \alpha_1), \dots, N_n(\Gamma_r \Rightarrow \alpha_r)$ , and  $\Gamma \Rightarrow \alpha$  can be inferred from  $\Gamma_1 \Rightarrow \alpha_1, \dots, \Gamma_r \Rightarrow \alpha_r$  by another instance of the same rule.

For illustration, we consider ( $\rightarrow$ L). Then,  $P_n(\beta) = \gamma \rightarrow \delta$  and the premises are  $P_n(\Gamma_1), \delta, P_n(\Gamma_2) \Rightarrow N_n(\alpha), P_n(\Phi) \Rightarrow \gamma$ , where we have  $P_n(\Gamma) = P_n(\Gamma_1), P_n(\Phi), \gamma \rightarrow \delta, P_n(\Gamma_2)$ . Here  $n$  is such that  $p(n) = i$ , and the rule introducing  $N_n(\Gamma \Rightarrow \alpha)$  has the fixed form. Clearly,  $\beta = \gamma' \rightarrow \delta', \gamma = N_n(\gamma'), \delta = P_n(\delta')$ . So, the premises are  $N_n(\Gamma_1, \delta', \Gamma_2 \Rightarrow \alpha), N_n(\Phi \Rightarrow \gamma')$ , and  $\Gamma \Rightarrow \alpha$  can be inferred from  $\Gamma_1, \delta', \Gamma_2 \Rightarrow \alpha$  and  $\Phi \Rightarrow \gamma'$ , by ( $\rightarrow$ L) (of the same form, actually).

Since  $N_n(\Gamma_j \Rightarrow \alpha_j)$  are provable for infinitely many  $n$ , they are provable for all  $n \in \omega$ , by (L4). Consequently,  $\Gamma_j \Rightarrow \alpha_j$ , for  $j = 1, \dots, r$ , are provable, by the induction hypothesis, which yields the provability of  $\Gamma \Rightarrow \alpha$ . ■

We prove a companion theorem.  $a(\Gamma \Rightarrow \alpha)$  denotes the number of occurrences of atoms in  $\Gamma \Rightarrow \alpha$ .

**THEOREM 2.** *Let  $\Gamma \Rightarrow \gamma$  contain no negative occurrence of  $*$ . Then,  $ACT\omega \vdash \Gamma \Rightarrow \gamma$  if and only if  $FL \vdash P_l(\Gamma \Rightarrow \gamma)$ , for  $l = a(\Gamma \Rightarrow \gamma)$ .*

**PROOF.** The ‘if’ part follows from (L2). We prove the ‘only if’ part. Assume  $ACT\omega \vdash \Gamma \Rightarrow \gamma$ . (\*L) introduces negative occurrences of  $*$ , so  $\Gamma \Rightarrow \gamma$  is provable in  $ACT^-$  which is  $ACT\omega$  without (\*L) ( $ACT^-$  is weaker than  $ACT$ ). We proceed by induction on proofs in  $ACT^-$ , with (Id) restricted to variables. If  $\Gamma \Rightarrow \gamma$  is (0L), then  $P_l(\Gamma \Rightarrow \gamma)$  is (0L), and so for (1R), (Id). If  $\Gamma \Rightarrow \gamma$  is (\*R1), then  $P_l(\Gamma \Rightarrow \gamma)$  equals  $P_l(\gamma)$  and is provable in FL, by (1R), ( $\vee$ R). We consider the rules. Let  $\Gamma \Rightarrow \gamma$  be inferred by ( $\vee$ L). Then,



the premises are  $\Gamma', \alpha, \Delta \Rightarrow \gamma$ ,  $\Gamma', \beta, \Delta \Rightarrow \gamma$ , and  $\Gamma = \Gamma', \alpha \vee \beta, \Delta$ . Denote the premises  $S', S''$  and the conclusion  $S$ . By the induction hypothesis,  $\text{FL} \vdash P_{l'}(S')$ ,  $\text{FL} \vdash P_{l''}(S'')$ , where  $l' = a(S')$ ,  $l'' = a(S'')$ . Clearly,  $l \geq \max(l', l'')$ . By (L3),  $\text{FL} \vdash P_l(S')$ ,  $\text{FL} \vdash P_l(S'')$ . (In the scope of \*-free sequents, FL and  $\text{ACT}\omega$  are equivalent). Since  $N_l$  distributes over  $\vee$  (see (4)), then  $\text{FL} \vdash P_l(S)$ , by ( $\vee$ L). All rules of FL are treated in a similar way.

We consider (\*R2). The premises are  $\Gamma_i \Rightarrow \alpha$ ,  $i = 1, \dots, n$ ,  $\Gamma = \Gamma_1, \dots, \Gamma_n$ , and  $\gamma = \alpha^*$ . Again,  $S_1, \dots, S_n$  denote the premises,  $S$  denotes the conclusion,  $l_i = a(S_i)$ ,  $l = a(S)$ . By the induction hypothesis,  $\text{FL} \vdash P_{l_i}(S_i)$ , and we have  $l_i \leq l$ , for all  $i = 1, \dots, n$ . Then,  $\text{FL} \vdash P_l(S_i)$ , for  $i = 1, \dots, n$ . By ( $\cdot$ R),  $\text{FL} \vdash N_l(\Gamma) \Rightarrow (P_l(\alpha))^n$ . Since  $n \leq l$ , then  $\text{FL} \vdash N_l(\Gamma) \Rightarrow (P_l(\alpha))^{\leq l}$ , by ( $\vee$ R), which yields  $\text{FL} \vdash P_l(S)$ . ■

As a consequence, we obtain a strengthening of Theorem 1. For any sequent  $S$ , we define  $f(S, n) = a(N_n(S))$ .

**THEOREM 3.**  $\text{ACT}\omega \vdash S$  if and only if, for all  $n \in \omega$ ,  $\text{FL} \vdash P_{f(S,n)}(N_n(S))$ .

$P_n$ ,  $N_n$  (with varying  $n$ ),  $f$  are computable, and FL is decidable, so  $\text{ACT}\omega$  is  $\Pi_1^0$ . This also follows from Theorem 1, since  $\text{ACT}^-$  is decidable, and  $\text{ACT}\omega \vdash N_n(S)$  if and only if  $\text{ACT}^- \vdash N_n(S)$ . For the KA-fragment of  $\text{ACT}\omega$ , there are no rules, moving formulas from the antecedent to the consequent, whence one can take  $l = a(\Gamma)$  in Theorem 2 and appropriately modify Theorem 3. Since this fragment amounts to  $\text{Eq}(\text{KA})$ , the above theorems for it can also be inferred from the Kozen theorem. No analogue of the Kozen theorem holds for equations with residuals. The equation  $a \rightarrow \bigvee X = \bigvee \{a \rightarrow x : x \in X\}$  is not valid in  $\text{ACTL}^*$ , whence we do not know whether the above theorems admit model-theoretic proofs (our proofs essentially use cut elimination).

### 3. Algebras of relations on finite universes

In this section, REL denotes the class of action lattices  $P(U^2)$ , consisting of all binary relations on  $U$ , with  $R \cdot S = R \circ S$ ,  $R \vee S = R \cup S$ ,  $R \wedge S = R \cap S$ ,  $1 = I_U$ ,  $0 = \emptyset$ ,  $R^* = \bigcup_{n \in \omega} R^n$ ,  $R \rightarrow S$ ,  $S \leftarrow R$  defined as in section 2. According to the terminology of relation algebras, REL is the class of square action lattices. FREL denotes the class of square action lattices  $P(U^2)$  such that  $U$  is a finite set.  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  is valid in REL, but not in  $\text{ACTL}^*$ , whence  $\text{Eq}(\text{ACTL}^*)$  is different from  $\text{Eq}(\text{REL})$  and  $\text{Eq}(\text{FREL})$ ; see [3] for examples without  $\wedge, \vee$ .

We prove that  $\text{Eq}(\text{FREL})$  is  $\Pi_1^0$ -hard, using the reduction applied in the proof of  $\Pi_1^0$ -hardness of  $\text{ACT}\omega$  in [4].  $\text{Eq}(\text{FREL})$  is evidently  $\Pi_1^0$ , whence it is  $\Pi_1^0$ -complete. We do not know whether  $\text{Eq}(\text{REL})$  is  $\Pi_1^0$ ; it is  $\Pi_1^0$ -hard, by the same reduction, but the proof given in [3] does not yield our present result.

*Types* are formulas of the  $(\leftarrow)$ -fragment of FL. (We could use  $\rightarrow$  as well.) A (classical, right-directed) *categorial grammar* is a triple  $G = (\Sigma, I_G, s)$  such that  $\Sigma$  is a finite alphabet,  $I_G$  is a finite relation between symbols from  $\Sigma$  and types, and  $s$  is a designated variable. One says that  $G$  assigns type  $\alpha$  to a string  $v_1 \dots v_n$ , ( $v_i \in \Sigma$ ), if there are types  $\alpha_1, \dots, \alpha_n$  such that  $(v_i, \alpha_i) \in I_G$ , for all  $i = 1, \dots, n$ , and  $\text{AB} \vdash \alpha_1, \dots, \alpha_n \Rightarrow \alpha$ .  $\text{AB}$  denotes the reduction system of Ajdukiewicz and Bar-Hillel, based on the reduction rule:  $\beta \leftarrow \alpha, \alpha \Rightarrow \beta$ . It can be axiomatized by (Id) and  $(\leftarrow\text{L})$ , restricted to  $(\leftarrow)$ -sequents. *The language of  $G$  ( $L(G)$ )* consists of all  $x \in \Sigma^+$  such that  $G$  assigns  $s$  to  $x$ .

The problem whether  $L(G) = \Sigma^*$ , for CF-grammars  $G$ , is  $\Pi_1^0$ -complete, and so for the problem whether  $L(G) = \Sigma^+$ , for  $\epsilon$ -free CF-grammars  $G$  [4]. A classical theorem of [1] states that, for any  $\epsilon$ -free CF-grammar  $G$ , one can construct a categorial grammar  $G'$  (with the same alphabet  $\Sigma$ ) such that  $L(G) = L(G')$  and  $I_{G'}$  employs only types of the form  $p, p \leftarrow q, (p \leftarrow q) \leftarrow r$ , where  $p, q, r$  are variables. Consequently, the problem whether  $L(G) = \Sigma^+$ , for categorial grammars  $G$ , is  $\Pi_1^0$ -complete (it is  $\Pi_1^0$ , since  $\text{AB}$  is decidable). This remains true even for categorial grammars with types restricted as above.

Let  $G = (\Sigma, I_G, s)$  be a categorial grammar with types of the above form. Let  $\Sigma = \{a_1, \dots, a_k\}$ . Let  $\alpha_1^i, \dots, \alpha_{n(i)}^i$  be all types  $\alpha$  such that  $(a_i, \alpha) \in I_G$ . Without loss of generality, we assume  $n(i) \neq 0$ , for  $i = 1, \dots, k$ . Set  $\beta_i = \alpha_1^i \wedge \dots \wedge \alpha_{n(i)}^i$ ,  $\gamma(G) = \beta_1 \vee \dots \vee \beta_k$ . The  $\Pi_1^0$ -hardness of  $\text{ACT}\omega$  follows from the equivalence, proved in [4]:

$$(\text{RED1}) \quad L(G) = \Sigma^+ \text{ iff } \text{ACT}\omega \vdash (\gamma(G))^*, \gamma(G) \Rightarrow s.$$

The family of all regular languages on  $\Sigma$  is a subalgebra of the complete action lattice  $P(\Sigma^*)$ , so it is a \*-continuous action lattice (regular languages are effectively closed under  $\rightarrow, \leftarrow, \wedge$ ).  $\text{REG}$  denotes the class of action lattices of regular languages on finite alphabets. In [3], the  $(\wedge, \rightarrow, \leftarrow)$ -fragment of FL is shown to be complete with respect to  $\text{REG}$ , which yields:

$$(\text{RED2}) \quad L(G) = \Sigma^+ \text{ iff } (\gamma(G))^*, \gamma(G) \Rightarrow s \text{ is valid in } \text{REG}.$$

**PROOF.** For the sake of completeness, we recall the proof of (RED2). By  $(*\text{L})$  and the provability of  $\alpha^n \Rightarrow \alpha^*$ , one proves that  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is

provable in  $\text{ACT}\omega$  iff, for all  $n \geq 1$ ,  $(\gamma(G))^n \Rightarrow s$  is so. The latter holds iff, for all  $n \geq 1$ , and all sequences  $(i_1, \dots, i_n) \in [k]^n$ ,  $\beta_{i_1}, \dots, \beta_{i_n} \Rightarrow s$  is provable. Here  $[k] = \{1, \dots, k\}$ . The same is true with ‘valid in  $\text{REG}$ ’ instead of ‘provable in  $\text{ACT}\omega$ ’.

We prove (RED2). Assume  $L(G) = \Sigma^+$ . Then,  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is provable in  $\text{ACT}\omega$ , by (RED1), whence the sequent is valid in  $\text{REG}$ , since  $\text{REG} \subseteq \text{ACTL}^*$ . Assume  $L(G) \neq \Sigma^+$ . By (RED1),  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is not provable in  $\text{ACT}\omega$ . By the above paragraph, there is a sequence  $(i_1, \dots, i_n) \in [k]^n$ , for some  $n \geq 1$ , such that  $\beta_{i_1}, \dots, \beta_{i_n} \Rightarrow s$  is not provable in  $\text{ACT}\omega$ , whence also in  $\text{FL}$ . This sequent belongs to the  $(\wedge, \rightarrow, \leftarrow)$ -fragment of  $\text{FL}$ , whence it is not valid in  $\text{REG}$ . By the above paragraph,  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is not valid in  $\text{REG}$ . ■

Let  $\mathcal{M} = (M, \cdot, 1)$  be a monoid. The powerset  $P(M)$  is a complete action lattice with  $X \vee Y = X \cup Y$ ,  $X \wedge Y = X \cap Y$ ,  $X \cdot Y = \{ab : a \in X, b \in Y\}$ ,  $X^* = \bigcup_{n \in \omega} X^n$ ,  $0 = \emptyset$ ,  $\mathbf{1} = \{1\}$ , and  $\rightarrow, \leftarrow$  defined as for languages. We denote this action lattice  $P(\mathcal{M})$ . PFM denotes the class of all action lattices  $P(\mathcal{M})$ , for finite monoids  $\mathcal{M}$ .

LEMMA 2. *Eq(PFM) is strictly contained in Eq(REG).*

PROOF. We prove  $\text{Eq}(\text{PFM}) \subseteq \text{Eq}(\text{REG})$ . Assume that  $\Gamma \Rightarrow \alpha$  is not valid in  $\text{REG}$ . Let  $p_1, \dots, p_n$  be all variables occurring in  $\Gamma \Rightarrow \alpha$ . There exist regular languages  $L_1, \dots, L_n$  on some finite alphabet  $\Sigma$  such that  $f(\Gamma) \not\subseteq f(\alpha)$ , for the assignment  $f(p_i) = L_i$ ,  $i = 1, \dots, n$ . Let  $\gamma_1, \dots, \gamma_k$  be all subformulas of formulas appearing in  $\Gamma \Rightarrow \alpha$ . Each language  $f(\gamma_j)$  is regular, so it determines a finite index congruence  $\sim_j$  on  $\Sigma^*$ , compatible with  $L(\gamma_j)$ . (A congruence  $\sim$  is compatible with  $L$ , if  $x \sim y$  entails:  $x \in L$  iff  $y \in L$ .) We define:  $x \sim y$  iff, for all  $j = 1, \dots, k$ ,  $x \sim_j y$ , ( $x, y \in \Sigma^*$ ). Clearly,  $\sim$  is a finite index congruence on  $\Sigma^*$ , compatible with each language  $L(\gamma_j)$ . We consider the monoid  $\Sigma^*/\sim$  and the action lattice  $P(\Sigma^*/\sim)$ ; it belongs to PFM. We define an assignment  $g$  in  $P(\Sigma^*/\sim)$ :  $g(p_i) = \{[x] : x \in L_i\}$ ,  $i = 1, \dots, n$ . By formula induction, we prove:  $x \in f(\gamma_j)$  iff  $[x] \in g(\gamma_j)$ , for  $j = 1, \dots, k$ . There is  $x \in f(\Gamma)$  such that  $x \notin f(\alpha)$ . Then,  $[x] \in g(\Gamma)$ ,  $[x] \notin g(\alpha)$ , whence  $\Gamma \Rightarrow \alpha$  is not valid in PFM.

We prove  $\text{Eq}(\text{PFM}) \neq \text{Eq}(\text{REG})$ . First, we observe that Horn formulas are equivalent to certain equations with respect the truth in models  $P(\mathcal{M})$ ,  $\mathcal{M}$  is a monoid.  $\alpha \leq \beta$  is equivalent to  $1 \leq \alpha \rightarrow \beta$ . The conjunction of  $1 \leq \alpha_i$ ,  $i = 1, \dots, n$ , is equivalent to  $1 \leq \alpha_1 \wedge \dots \wedge \alpha_n$ . The implication ‘if  $1 \leq \alpha$  then  $1 \leq \beta$ ’ is equivalent to  $1 \wedge \alpha \leq \beta$ . Now, ‘if  $1 \leq aa$  then  $1 \leq a$ ’ is valid in  $\text{REG}$ , but not in PFM. To prove the latter, consider a finite, nontrivial

group  $\mathcal{M}$ . For  $x \in M$ ,  $x \neq 1$ , we have  $x^{-1} \neq 1$ . An assignment  $f$  in  $P(\mathcal{M})$  is defined by  $f(a) = \{x, x^{-1}\}$ . Then,  $1 \in f(aa)$ ,  $1 \notin f(a)$ . Consequently, the corresponding equation  $1 \wedge (aa) \leq a$  is valid in REG, but not in PFM. ■

(RED3)  $L(G) = \Sigma^+$  iff  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is valid in PFM.

PROOF. The ‘only if’ part follows from (RED1), since  $\text{PFM} \subseteq \text{ACTL}^*$ . The ‘if’ part follows from (RED2) and Lemma 2. ■

THEOREM 4. *Eq(PFM) is  $\Pi_1^0$ -complete.*

PROOF. By (RED3), Eq(PFM) is  $\Pi_1^0$ -hard. Obviously, it is  $\Pi_1^0$ . ■

(RED4)  $L(G) = \Sigma^+$  iff  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is valid in FREL.

PROOF. The ‘only if’ part follows from (RED1), as above. To prove the ‘if’ part, assume  $L(G) \neq \Sigma^+$ . We denote  $\gamma = (\gamma(G))^* \cdot \gamma(G)$ . By (RED3),  $\gamma \Rightarrow s$  is not valid in PFM. So, there exist a finite monoid  $\mathcal{M}$  and an assignment  $f$  in  $P(\mathcal{M})$  such that  $f(\gamma) \not\subseteq f(s)$ . We consider the finite relational action lattice  $P(M^2)$ . For any formula  $\alpha$ , we define a relation  $R(\alpha) \subseteq M^2$ :  $R(\alpha) = \{(x, xy) \in M^2 : x \in M, y \in f(\alpha)\}$ . One proves:

$$R(\alpha \cdot \beta) = R(\alpha) \circ R(\beta), \quad R(\alpha \vee \beta) = R(\alpha) \cup R(\beta), \quad R(\alpha^*) = (R(\alpha))^*, \quad (10)$$

$$R(0) = \emptyset, \quad R(1) = I_M, \quad (11)$$

$$R(\alpha \wedge \beta) \subseteq R(\alpha) \cap R(\beta), \quad R(\alpha \leftarrow \beta) \subseteq R(\alpha) \leftarrow R(\beta). \quad (12)$$

We define an assignment  $g$  in  $P(M^2)$ :  $g(p) = R(p)$ , for any variable  $p$ . By the special form of types  $\alpha_j^i$ , involved in  $G$ , we have  $R(\alpha_j^i) \subseteq g(\alpha_j^i)$  (we use the second inclusion (12) and:  $Y \leftarrow X \subseteq Z \leftarrow X$  if  $Y \subseteq Z$ ),  $R(\beta_i) \subseteq g(\beta_i)$  (we use the first inclusion (12)),  $R(\gamma(G)) \subseteq g(\gamma(G))$ ,  $R((\gamma(G))^*) \subseteq g((\gamma(G))^*)$ , and  $R(\gamma) \subseteq g(\gamma)$  (we use equations (10) and the monotonicity of  $\cup, *, \circ$ ). There is  $y \in M$  such that  $y \in f(\gamma)$ ,  $y \notin f(s)$ . Then,  $(1, y) \in R(\gamma)$ ,  $(1, y) \notin R(s)$ , whence  $(1, y) \in g(\gamma)$ ,  $(1, y) \notin g(s)$ . We have shown that  $(\gamma(G))^*, \gamma(G) \Rightarrow s$  is not valid in FREL. ■

THEOREM 5. *Eq(FREL) is  $\Pi_1^0$ -complete.*

PROOF. The proof is similar to that of Theorem 4. ■

The sequent  $p, 1 \leftarrow p \Rightarrow 1$  is not valid in FREL [3]. Take  $U = \{x, y\}$ ,  $f(p) = \{(x, y)\}$ ; then,  $f(1 \leftarrow p) = \{(x, y), (y, x), (y, y)\}$ ,  $f(p \cdot (1 \leftarrow p)) = \{(x, x), (x, y)\}$ . We show that this sequent is valid in PFM, whence in REG. Let  $\mathcal{M}$  be a finite monoid. Let  $f$  be an assignment in  $P(\mathcal{M})$ .  $f(1 \leftarrow p)$  consists of all  $x \in M$  such that  $xy = 1$ , for all  $y \in f(p)$ . In finite monoids,  $xy = 1$  entails  $yx = 1$  (represent the monoid as a monoid of functions; for functions on a finite set, every surjection is an injection, and conversely), so  $f(p \cdot (1 \leftarrow p)) \subseteq \{1\}$ . Another example: the equation corresponding to (K2)  $1 \wedge ((ba) \rightarrow b) \leq (ba^*) \rightarrow b$  is valid in PFM, but not in FREL. Take  $U = \{x, y, z, u\}$ ,  $f(a) = \{(y, z), (z, u)\}$ ,  $f(b) = \{(x, y), (u, u)\}$ ; so  $f(ba) = \{(x, z)\}$ ,  $f(a^2) = \{(y, u)\}$ ,  $f(ba^2) = \{(x, u)\}$ , and  $(u, u) \in f((ba) \rightarrow b)$ ,  $(u, u) \notin f((ba^2) \rightarrow b)$ , which yields  $(u, u) \notin f((ba^*) \rightarrow b)$ . We have shown  $\text{Eq}(\text{FREL}) \neq \text{Eq}(\text{PFM})$ ,  $\text{Eq}(\text{FREL}) \neq \text{Eq}(\text{REG})$ . Also  $\text{Eq}(\text{ACTL}^*)$  is strictly contained in  $\text{Eq}(\text{REG})$ ,  $\text{Eq}(\text{PFM})$  and  $\text{Eq}(\text{FREL})$ ; for each of the latter theories, we have found out an equation which belongs to this theory, but not to  $\text{Eq}(\text{ACTL}^*)$ .

REMARKS. (1) The  $\Pi_1^0$ -hardness of  $\text{Eq}(\text{REL})$ , proved in [3], directly follows from (RED4) and the fact that  $\text{REL} \subseteq \text{ACTL}^*$ . On the other hand, analogous results for  $\text{Eq}(\text{ACTA}^*)$  and  $\text{Eq}(\text{REL})$  without  $\wedge$ , obtained in [4, 3], cannot be directly adapted to FREL, since they employ a grammar  $G$  with more complicated types. (2) Similar complexity bounds can be found for  $\text{ACT}\omega^+$  without 1 and with the restricted Kleene closure  $^+$  instead of  $^*$ , and the corresponding equational theories of  $\epsilon$ -free regular languages and algebras of subrelations of an irreflexive and transitive (finite) relation. (3) It follows from these results that no effectively axiomatizable dynamic logic, in which program constructions contain regular operations and residuals (as suggested in [8, 21]), is complete with respect to standard relational frames. (4) The Horn theories of  $\text{ACTA}^*$ ,  $\text{ACTL}^*$ ,  $\text{REL}$  are  $\Pi_1^1$ -complete, since their Kleene algebra versions are of this complexity [12, 7], and every  $^*$ -continuous Kleene algebra can be embedded into a complete action lattice [4]. By (RED2), (RED4), the Horn theories of  $\text{REG}$  and  $\text{FREL}$  are  $\Pi_1^0$ -complete. If complementation were added to Kleene algebras in  $\text{REL}$ , then even the equational theory of this class would be  $\Pi_1^1$ -complete (it is well-known that with product and boolean operations, Horn formulas valid in  $\text{REL}$  can be represented by equations), and this remains true for action algebras. (5) The equational theory of  $^*$ -continuous distributive action lattices (i.e. action lattices whose lattice reducts are distributive) is  $\Pi_1^0$ -hard; it is a consequence of our results on the  $\Pi_1^0$ -hardness of  $\text{Eq}(\text{REL})$  (also of (RED4)). Hopefully, the upper bound  $\Pi_1^0$  can also be proved, using  $^*$ -elimination for modified sequent systems with two commas: one corresponding to product

(catenation) and one to conjunction  $\wedge$ . Such systems for relevant logics were designed by Mints [15] and Dunn [5, 6].

#### 4. Categorical grammars

Categorical grammars or type-logical grammars are formal grammars which assign types to lexical atoms (words) and employ a type logic to process syntactic structures in accordance with their semantical interpretation. In general, types are formed out of not only  $\leftarrow$ , as in section 3, but also  $\rightarrow$ ,  $\cdot$  and possibly  $\wedge$ ,  $\vee$  and other operations. The simple system AB can be replaced by a richer logic, e.g. the Lambek calculus (associative or not) or another fragment of FL. The notion of a categorical grammar is defined as in section 3 except that a different logic may be involved, and an arbitrary formula may replace the designated variable  $s$ . An overview of this discipline and literature references can be found in [2, 16].

It seems reasonable to propose action logic as a type logic for categorical grammars. Let us recall that type  $\alpha \rightarrow \beta$  (resp.  $\beta \leftarrow \alpha$ ) is interpreted as a type of a functor expression  $x$  such that, for any (argument) expression  $y$  of type  $\alpha$ ,  $yx$  (resp.  $xy$ ) is of type  $\beta$ . Let  $s, n_1, n$  be atomic types of sentences, noun phrases, and common nouns, respectively. Then,  $n_1 \rightarrow s$  is a type of verb phrases,  $(n_1 \rightarrow s) \leftarrow n_1$  of transitive verb phrases,  $n_1 \leftarrow n$  of determiners,  $n \leftarrow n$  of adjectives,  $(n_1 \rightarrow s) \rightarrow (n_1 \rightarrow s)$  of prepositional phrases, and so on. Alternatively, one might introduce an atomic type  $a$  for adjectives and assign  $a^* \rightarrow n$  to common nouns. With more subtlety, different nouns can be assigned different types of the form  $(a_1 \vee \dots \vee a_k)^* \rightarrow n$ , where  $a_1, \dots, a_k$  are atomic types of some subcategories of adjectives, which may be connected with the noun. (In this way, one could eliminate Chomsky's 'colorless green ideas'.) To regard finer constraints, for instance, the preferred precedence of some parts of the list of adjectives, more complicated regular expressions are to be used as argument types. For instance, 'tall American student' seems more plausible than 'American tall student'. Then, 'student' will be assigned type  $((a_R)^*(a_A)^*) \rightarrow n$ , where  $a_R$  is the type of relative adjectives and  $a_A$  of absolute adjectives.

The general idea is to marry type logic with regular algebra. We have no place here to develop this program in detail. Since  $\text{ACT}\omega$  is not recursively enumerable, effective grammars must employ its subsystems. As suggested above, \*-types naturally appear on argument positions of types in antecedents of sequents, and these are positive occurrences in sequents. With this constraint,  $\text{ACT}^-$  is the complete logic; it is decidable.  $\text{ACT}^-$  can be extended to other decidable subsystems of ACT. It is possible that the

full ACT is decidable, and similarly for the multiplicative (i.e.  $\vee, \wedge$ -free) fragment of  $\text{ACT}\omega$ . (The undecidability proof from [4] needs either  $\wedge$ , or  $\vee$ , to handle several types assigned to one symbol; for one-valued categorial grammars, based on the  $(\leftarrow)$ -fragment of AB, the total language problem is decidable, since they are effectively equivalent to some deterministic push-down automata.) The multiplicative  $\text{ACT}^-$  is NP-complete, since its conservative fragment  $L^*$  is NP-complete [20].

As for other kinds of categorial grammars, we are faced with problems of generative capacity. We know only a little about classes of languages generated by grammars involving  $*$ -types. (For  $*$ -free types, multiplicative systems yield context-free languages, and with  $\wedge, \vee$  or other extras they fall in the class of mildly context-sensitive languages; see [2, 16] for discussion.) Since  $\text{ACT}\omega$  is  $\Pi_1^0$ -complete, the universal membership problem for the class of all categorial grammars based on  $\text{ACT}\omega$  is not recursively enumerable. By the Post theorem, no grammar from this class can generate a nonrecursive, recursively enumerable language.

In order to initiate the study of generative capacity of classes of categorial grammars, based on certain fragments of  $\text{ACT}\omega$ , admitting  $*$ -types, we provide two sample results of that kind. We do not recall standard notions of the theory of finite state automata. DFSA abbreviates ‘deterministic finite state automaton’.

**PROPOSITION 1.** *Grammars based on the Kleene fragment of  $\text{ACT}\omega$  generate precisely all regular languages (arbitrary regular expressions can be taken for  $s$ ).*

**PROOF.** For a finite set  $X$  of regular expressions on  $\Sigma$  and a regular expression  $\alpha$  on  $\Sigma$ , by  $L(X, \alpha)$  we denote the set of all  $\Gamma \in X^*$  such that  $\text{ACT}\omega \vdash \Gamma \Rightarrow \alpha$ . We prove that  $L(X, \alpha)$  is a regular language (on alphabet  $X$ ), by induction on  $c(X)$  equal to the sum of complexities of all expressions in  $X$  (by the complexity of a regular expression we mean the total number of operation symbols and constants 0, 1 occurring in it). If  $c(X) = 0$ , then  $X$  consists of symbols from  $\Sigma$  only, so  $L(X, \alpha) = L(\alpha) \cap X^*$  is a regular language (we use the Kozen theorem and the fact that the Kleene fragment of  $\text{ACT}\omega$  amounts to  $\text{Eq(KA)}$ ). Assume  $c(X) > 0$ . Then,  $X$  must contain an expression  $\beta$  whose complexity is greater than 0. We consider several cases.

$\beta = 0$ . Set  $Y = X - \{0\}$ . We have  $c(Y) < c(X)$ . By the induction hypothesis,  $L(Y, \alpha)$  is regular. There exists a DFSA  $\mathcal{A}$  such that  $L(\mathcal{A}) = L(Y, \alpha)$ . The transition function  $\delta$  of  $\mathcal{A}$  is extended by  $\delta(q, 0) = f$ , for any state  $q$  of  $\mathcal{A}$ , and  $\delta(f, \gamma) = f$ , for any  $\gamma \in X$ , where  $f$  is a new final state. The new DFSA accepts  $L(X, \alpha)$ .

$\beta = 1$ . Set  $Y = X - \{1\}$ . Again, there exists a DFSA  $\mathcal{A}$  which accepts  $L(Y, \alpha)$ . Its transition function  $\delta$  is extended by  $\delta(q, 1) = q$ , for any state  $q$  of  $\mathcal{A}$ .

$\beta = \beta_1\beta_2$ . Set  $Y = (X - \{\beta\}) \cup \{\beta_1, \beta_2\}$ . The DFSA accepting  $L(Y, \alpha)$  is modified, by setting  $\delta(q, \beta) = \delta(\delta(q, \beta_1), \beta_2)$ .

$\beta = \beta_1 \vee \beta_2$ .  $Y$  is defined as above. From the DFSA  $\mathcal{A}$  which accepts  $L(Y, \alpha)$  we construct a new DFSA  $\mathcal{A}'$  whose states are nonempty sets of states of  $\mathcal{A}$ . For a set of states  $P$ , we define  $\delta'(P, \gamma) = \{\delta(q, \gamma) : q \in P\}$ , if  $\gamma \in Y$ , and  $\delta'(P, \beta) = \delta'(P, \beta_1) \cup \delta'(P, \beta_2)$ . The initial state of  $\mathcal{A}'$  is  $\{q_0\}$ , where  $q_0$  is the initial state of  $\mathcal{A}$ . The set of final states of  $\mathcal{A}'$  consists of all nonempty sets of final states of  $\mathcal{A}$ . We show that  $\mathcal{A}'$  accepts  $L(X, \alpha)$ . First, the sequent:

$$\Gamma_0, \beta, \Gamma_1, \beta, \dots, \beta, \Gamma_n \Rightarrow \alpha, \quad (13)$$

where  $\Gamma_i \in (X - \{\beta\})^*$ , for  $i = 0, \dots, n$ , is provable in  $\text{ACT}\omega$  if and only if, for all sequences  $(i_1, \dots, i_n) \in [2]^n$ , the sequent:

$$\Gamma_0, \beta_{i_1}, \Gamma_1, \beta_{i_2}, \dots, \beta_{i_n}, \Gamma_n \Rightarrow \alpha \quad (14)$$

is provable (use the distribution of product under  $\vee$  in Kleene algebras). It is easy to see that  $\mathcal{A}'$  accepts the antecedent of (13) if and only if  $\mathcal{A}$  accepts the antecedent of (14), for all  $(i_1, \dots, i_n) \in [2]^n$ .

$\beta = \gamma^*$ . The construction is similar to that from the preceding case. We set  $\delta'(P, \beta) = \bigcup_{n \in \omega} \delta'(P, \gamma^n)$ .

It follows that, for any grammar  $G = (\Sigma, I_G, \alpha)$  such that  $I_G$  involves regular expressions (on a finite set  $P$  of variables) in the role of types, the language  $L(G)$  is regular (we use the fact that regular languages are closed under substitution). Conversely, every regular language can be obtained in this way; one constructs a trivial grammar, as for  $c(X) = 0$  above. ■

Types of the form  $\alpha_1 \rightarrow \dots \rightarrow \alpha_m \rightarrow p \leftarrow \beta_1 \leftarrow \dots \leftarrow \beta_n$ , where  $p$  is a variable, and  $\alpha_i, \beta_j$  are regular expressions (parentheses grouped to the middle), will be called *types with regular arguments*.

**PROPOSITION 2.** *Categorical grammars of the form  $G = (\Sigma, I_G, s)$  such that  $I_G$  involves types with regular arguments only and  $s$  is a variable generate precisely  $\epsilon$ -free context-free languages.*

By the restricted form of types, the complete logic is  $\text{ACT}^-$ . The fact that every  $\epsilon$ -free context-free language can be obtained in this way follows from the result in [1], discussed in section 3: for types of the form  $p, p \leftarrow q, (p \leftarrow q) \leftarrow r$ , AB and  $\text{ACT}\omega$  yield the same provable sequents  $\Gamma \Rightarrow s$ . We



omit a more subtle proof of the converse direction; it employs Theorem 2 and other proof-theoretic properties of  $\text{ACT}^-$ .

## References

- [1] BAR-HILLEL, Y., C. GAIFMAN, and E. SHAMIR, ‘On categorial and phrase structure grammars’, *Bull. Res. Council Israel* F9 (1960), 155–166.
- [2] BUSZKOWSKI, W., ‘Mathematical Linguistics and Proof Theory’, in J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*, Elsevier, The MIT Press, Amsterdam, Cambridge MA., 1997, pp. 683–736.
- [3] BUSZKOWSKI, W., ‘On the complexity of the equational theory of relational action algebras’, in R. Schmidt (ed.), *Relations and Kleene Algebra in Computer Science*, LNCS 4136, 2006, pp. 106–119.
- [4] BUSZKOWSKI, W., ‘On action logic: Equational theories of action algebras’, *Journal of Logic and Computation* 17.1 (2007), 199–217.
- [5] DUNN, J.M., ‘A ”Gentzen system” for positive relevant implication’ (abstract), *Journal of Symbolic Logic* 38 (1973), 356–357.
- [6] DUNN, J.M., ‘Relevance Logic and Entailment’, in D. Gabbay and F. Guenther (eds.), *Handbook of Philosophical Logic*, vol. III, D. Reidel, Dordrecht, 1986, pp. 117–224.
- [7] HARDIN, C., and D. KOZEN, ‘On the complexity of the Horn theory of REL’, manuscript, Cornell University, 2003.
- [8] HOARE, C., and H. JIFENG, ‘The weakest prespecification’, *Fundamenta Informaticae* 9 (1986), 51–84, 217–252.
- [9] JIPSEN, P., ‘From semirings to residuated Kleene algebras’, *Studia Logica* 76 (2004), 291–303.
- [10] KOZEN, D., ‘On Kleene algebras and closed semirings’, *Proc. of MFCS 1990*, LNCS 452, 1990, pp. 26–47.
- [11] KOZEN, D., ‘A completeness theorem for Kleene algebras and the algebra of regular events’, *Information and Computation* 110:2 (1994), 366–390.
- [12] KOZEN, D., ‘On the complexity of reasoning in Kleene algebra’, *Information and Computation* 179 (2002), 152–162.
- [13] KOZEN, D., and F. SMITH, ‘Kleene algebra with tests: Completeness and decidability’, in D. van Dalen and M. Bezem (eds.), *Proc. 10th Int. Workshop Computer Science Logic (CSL’96)*, LNCS 1258, Springer, 1996, pp. 244–259.
- [14] LAMBEK, J., ‘The mathematics of sentence structure’, *American Mathematical Monthly* 65 (1958), 154–170.
- [15] MINTS, G., ‘Cut Elimination Theorem in Relevant Logics’, *Journal of Mathematical Sciences* 6 (1976), 422–428.
- [16] MOORTGAT, M., ‘Categorial Type Logics’, in J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*, Elsevier, The MIT Press, Amsterdam, Cambridge MA, 1997, pp. 93–177.
- [17] ONO, H., ‘Semantics of substructural logics’, in P. Schroeder-Heister and K. Dosen (eds.), *Substructural Logics*, Clarendon Press, Oxford, 1993, pp. 259–291.

- [18] PALKA, E., ‘An infinitary sequent system for the equational theory of \*-continuous action lattices’, *Fundamenta Informaticae* 78.2 (2007), 295–309.
- [19] PALKA, E., ‘Logical systems for Kleene algebras and action algebras’, PhD Thesis, 2007. In Polish, unpublished.
- [20] PENTUS, M., ‘Lambek calculus is NP-complete’, *Theoretical Computer Science* 357 (2006), 186–201.
- [21] PRATT, V., ‘Action logic and pure induction’, in J. van Eijck (ed.), *Logics in AI*, LNAI 478, 1991, pp. 97–120.

WOJCIECH BUSZKOWSKI  
Faculty of Mathematics and Computer Science  
Adam Mickiewicz University in Poznań  
Research Group on Mathematical Linguistics  
Rovira i Virgili University in Tarragona  
Poland, Spain  
buszko@amu.edu.pl

EWA PALKA  
Faculty of Mathematics and Computer Science  
Adam Mickiewicz University  
Poznań, Poland  
pewka@amu.edu.pl