Fundamenta Informaticae 78 (2007) 295–309
IOS Press

An infinitary sequent system for the equational theory of *-continuous action lattices

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Abstract. We present an infinitary logic ACT_ω in the form of a Gentzen-style sequent system, which is equivalent to the equational theory of *-continuous action lattices [9]. We prove the cut-elimination theorem for ACT_ω and, as a consequence, a theorem on the elimination of negative occurrences of *. This shows that ACT_ω is Π^0_1, whence, by a result of Buszkowski [1], it is Π^0_1-complete.

Keywords: Kleene algebra, action algebra, action lattice, cut-elimination, complexity

1. Introduction

Kozen [4] introduces Kleene algebras as an algebraic counterpart of regular algebras. The Kozen completeness theorem states that $\alpha = \beta$ is true for regular expressions iff $\alpha = \beta$ is valid in Kleene algebras. Kleene algebras do not form a variety, and it is known that the equations true for regular expressions cannot be axiomatized by any finite number of equations [10].

Pratt [9] defines action algebras as Kleene algebras with residuals. In the language without residuals, the equations valid in action algebras are the same as those valid in Kleene algebras. Action algebras form a finitely based variety; this is also true for action lattices, e.g action algebras admitting the meet operation $\wedge$.

On the other hand, many basic properties of action algebras are different from properties of Kleene algebras. While the equational theory of Kleene algebras is decidable (PSPACE-complete), the complexity of the equational theory of action algebras is not known [2], [1]. Every complete action algebra is

[*]The author thanks Prof. Wojciech Buszkowski for the supervision of this research and helpful suggestions.

*-continuous, which is not true for Kleene algebras [1]. The equational theory of *-continuous Kleene algebras equals the one of all Kleene algebras, whence it is decidable. It is shown in [1] that the equational theory of *-continuous action algebras (lattices) is Π^0_1-hard, whence it is not recursively enumerable and cannot be equal to the one of all action algebras (lattices). The latter does not possess FMP (finite model property), but the former possesses FMP [1]. In [8] it is shown that the equational theory of Kleene algebras possesses FMP.

In this paper we consider an axiomatization of the equational theory of *-continuous action lattices in the form of a Gentzen-style sequent system. This system extends Full Lambek Calculus (FL) in the sense of Ono [7], Jipsen [2], by some axioms and rules for *. (FL corresponds to the equational theory of residuated lattices). The left-introduction rule for * is an infinitary rule (rule (ω)); from the infinite set of premises $\Gamma_1, A^n, \Gamma_2 \Rightarrow B$, for all $n \in \omega$, one infers $\Gamma_1, A^*, \Gamma_2 \Rightarrow B$. Accordingly, the logic is an infinitary logic; we denote it by ACT_ω.

In section 3 we prove the cut-elimination theorem for ACT_ω, by a transfinite induction on ranks of provable sequents. This strengthens several results on cut-elimination for finitary systems contained in ACT_ω, e.g. FL [7] and the Lambek calculus [6].

In section 4 we prove that ACT_ω is complete with respect to the class of *-continuous action lattices. We also define and study syntatic operations $P_n, A_n, (n \in \omega)$, which transform any formula A into a formula $P_n(A)$ (resp. $N_n(A)$) without positive (resp. negative) occurrences of *. For a sequent $\Gamma \Rightarrow A$, $N_n(\Gamma \Rightarrow A)$ is defined as the sequent $P_n(\Gamma) \Rightarrow N_n(A)$; it contains no negative occurrences of *. Several lemmas show how are these operations related to axioms and inference rules of ACT_ω.

In section 5 we prove the main result of this paper: the *-elimination theorem. According to the theorem, $\Gamma \Rightarrow A$ is provable in ACT_ω iff, for all $n \in \omega$, $N_n(\Gamma \Rightarrow A)$ is provable in ACT_ω. Since $N_n(\Gamma \Rightarrow A)$ contains no negative occurrences of *, then it is provable in ACT_ω iff it is provable in ACT_ω without the rule (ω). The latter system is a finitary cut-free sequent system, which admits an effective proof-search procedure, and consequently, it is decidable. Then, ACT_ω is Π^0_1. Together with the Π^0_1-hardness of ACT_ω [1], this yields the Π^0_1-completeness of ACT_ω.

2. Preliminaries

This chapter presents some preliminaries. We define a Kleene algebra, an action lattice and a *-*continuous* action lattice.

A *Kleene algebra* [4] is an algebraic structure $\mathcal{A}$=$(A, +, \cdot, *, 0, 1)$ with two distinguished elements 0 and 1, two binary operations $+$ and $\cdot$, and a unary operation $*$ satisfying the following axioms.

$$a + (b + c) = (a + b) + c \tag{1}$$

$$a + b = b + a \tag{2}$$

$$a + 0 = a \tag{3}$$

$$a + a = a \tag{4}$$

$$a(bc) = (ab)c \tag{5}$$

$$1a = a \tag{6}$$

$$a1 = a \tag{7}$$

$$a(b + c) = ab + ac \tag{8}$$

$$(a + b)c = ac + bc \tag{9}$$

$$0a = 0 \tag{10}$$

$$a0 = 0 \tag{11}$$

$$1 + aa^* \leq a^* \tag{12}$$

$$1 + a^*a \leq a^* \tag{13}$$

$$\text{if } ax \leq x \text{ then } a^*x \leq x \tag{14}$$

$$\text{if } xa \leq x \text{ then } xa^* \leq x \tag{15}$$

where $\leq$ denotes the partial order on A, defined as follows:

$$a \leq b \Leftrightarrow a + b = b \tag{16}$$

Axioms (1)-(4) say that $(A, +, 0)$ is an idempotent commutative monoid, and axioms (5)-(7) say that $(A, \cdot, 1)$ is a monoid. Note that axioms (12)-(15) say, essentialy, that the operation $*$ behaves like the asterate operator on sets of strings or the reflexive transitive closure operator on binary relations. The following properties are true in all Kleene algebras:

$$1 \leq a^* \tag{17}$$

$$a \leq a^* \tag{18}$$

$$\text{if } a \leq b \text{ and } c \leq d \text{ then } ac \leq bd \tag{19}$$

$$a \leq x \text{ and } b \leq x \text{ iff } a + b \leq x \tag{20}$$

A standard Kleene algebra is *the algebra of languages* on a finite alphabet Σ. A *language* on Σ is a set of finite strings on Σ. The largest language on Σ is denoted Σ^*. The empty string is denoted ε. For $L, L_1, L_2 \subseteq \Sigma^*$, one sets: $L_1 \vee L_2 = L_1 \cup L_2$, $L_1 \cdot L_2 = \{xy : x \in L_1, y \in L_2\}$, $L^* = \bigcup_{n \in \omega} L^n$, $0 = \emptyset$, $1 = \{\varepsilon\}$, where $L^0 = 1, L^{n+1} = L^n L$. Another example is the algebra of all binary relations on a set U. Now, product is relational product, 1 is the identity relation, and the remaining notions are defined as above.

A Kleene algebra $\mathcal{A}$ is said to be *-*continuous*, if $xa^*y = sup\{xa^ny : n \in \omega\}$, for all $x, y, a \in A$. Clearly, the algebra of languages and the algebra of relations are *-*continuous*.

Regular expressions on Σ are variable-free terms of the (first-order) language of Kleene algebras enriched with all symbols from Σ as new individual constants. For $a \in \Sigma$, one sets $L(a) = \{a\}$ and extends the mapping L to a (unique) homomorphism from the (variable-free) term algebra to the algebra of languages on Σ. For a regular expression α, the language $L(\alpha)$ is called *the language denoted by* α. The equality $\alpha = \beta$ is said to be *true for regular expressions*, if $L(\alpha) = L(\beta)$.

Pratt [9] defines an *action algebra* as an algebra $\mathcal{A}$=$(A, +, \cdot, *, \rightarrow, \leftarrow, 0, 1)$ such that +,$\cdot$,$*$,0,1 are as above, and $\rightarrow$,$\leftarrow$ are binary operations on A, which satisfy axioms (1)-(7) and the following:

$$a \leq c \leftarrow b \text{ iff } ab \leq c \text{ iff } b \leq a \rightarrow c \tag{21}$$

$$1 + a^*a^* + a \leq a^* \tag{22}$$

$$\text{if } 1 + bb + a \leq b \text{ then } a^* \leq b \tag{23}$$

where the relation $\leq$ is as above. Operations $\rightarrow$ and $\leftarrow$ are called *the right residuation* and *the left residuation*, respectively. We use (RES) to denote the axiom (21). Pratt [9] shows that every action algebra is a Kleene algebra. An action algebra is **-continuous* iff $a^* = sup\{a^n : n \in \omega\}$. Pratt [9] proves that action algebras form a finitely based variety.

An action lattice is an action algebra which is a lattice, this means, it admits a *meet* operation $\wedge$, satisfying axioms (1)-(3) with $+$ replaced by $\wedge$ (called *the semilattice axioms*) and the absorption axioms $a + (a \wedge b) = a$, $a \wedge (a + b) = a$. Some results on action lattices can be found in Kozen [5].

The algebra of languages can be expanded to an action lattice by setting $L_1 \wedge L_2 = L_1 \cap L_2$ and:

$$L_1 \rightarrow L_2 = \{x \in \Sigma^* : L_1\{x\} \subseteq L_2\}$$
$$L_1 \leftarrow L_2 = \{x \in \Sigma^* : \{x\}L_2 \subseteq L_1\}.$$

Regular languages on Σ, i.e. languages denoted by regular expressions on Σ, form a subalgebra of this action lattice. The algebra of relations on U can also be expanded to an action lattice with the meet defined as the set-theoretic intersection and residuals defined as follows:

$$R_1 \rightarrow R_2 = \{(x, y) \in U^2 : R_1 \circ \{(x, y)\} \subseteq R_2\}$$
$$R_1 \leftarrow R_2 = \{(x, y) \in U^2 : \{(x, y)\} \circ R_2 \subseteq R_1\}.$$

We consider an infinitary logic ACT_ω which is complete with respect to *-continuous action lattices. This logic is formalized as a Gentzen-style sequent system. It amounts to an extension of Full Lambek Calculus (FL) in the sense of [7], [2] by axioms and inference rules concerning *. Atomic formulas of ACT_ω are variables and constants 0 and 1. Formulas of ACT_ω are formed out of atomic formulas by means of the connectives $*, ;, \rightarrow, \leftarrow, \vee, \wedge$. We use characters p, q for variables and A, B, C for formulas. Greek capitals Γ, Φ, Ψ represent finite strings of formulas. Sequents are expressions of the form $\Gamma \Rightarrow A$. The axioms of ACT_ω are:

$$\text{(Id) } A \Rightarrow A, \text{ (0L) } \Gamma_1, 0, \Gamma_2 \Rightarrow A, \text{ (1R) } \Rightarrow 1, \text{ (}*_1\text{R) } \Rightarrow A^*$$

and the inference rules are the following:

$$(1L)\ \frac{\Gamma_1, \Gamma_2 \Rightarrow A}{\Gamma_1, 1, \Gamma_2 \Rightarrow A},$$

$$(\wedge_1 L)\ \frac{\Gamma_1, A, \Gamma_2 \Rightarrow C}{\Gamma_1, A \wedge B, \Gamma_2 \Rightarrow C},\ (\wedge_2 L)\ \frac{\Gamma_1, B, \Gamma_2 \Rightarrow C}{\Gamma_1, A \wedge B, \Gamma_2 \Rightarrow C},$$

$$(\wedge R)\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B},$$

$$(\vee L)\ \frac{\Gamma_1, A, \Gamma_2 \Rightarrow C \quad \Gamma_1, B, \Gamma_2 \Rightarrow C}{\Gamma_1, (A \vee B), \Gamma_2 \Rightarrow C},$$

$$(\vee_1 R)\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B},\ (\vee_2 R)\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B},$$

$$(; L)\ \frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow C}{\Gamma_1, A; B, \Gamma_2 \Rightarrow C},\ (; R)\ \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A; B},$$

$$(\to R)\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B},\ (\leftarrow R)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B \leftarrow A},$$

$$(\to L)\ \frac{\Gamma \Rightarrow A \quad \Gamma_1, B, \Gamma_2 \Rightarrow C}{\Gamma_1, \Gamma, A \to B, \Gamma_2 \Rightarrow C}, (\leftarrow L) \frac{\Gamma \Rightarrow A \quad \Gamma_1, B, \Gamma_2 \Rightarrow C}{\Gamma_1, B \leftarrow A, \Gamma, \Gamma_2 \Rightarrow C},$$

$$(*_2 R)\ \frac{\Gamma_1 \Rightarrow A, \ldots, \Gamma_n \Rightarrow A}{\Gamma_1, \ldots, \Gamma_n \Rightarrow A^*} \text{ for any } n \geq 1,$$

$$(*L)\ \frac{(\Gamma_1, A^n, \Gamma_2 \Rightarrow B)_{n \in \omega}}{\Gamma_1, A^*, \Gamma_2 \Rightarrow B}.$$

($*_2$R) is an infinite family of finary rules. ($*$L) is an infinitary rule. We use (ω) to denote the rule ($*$L). Here A^n stands for the string of n copies of A, A^0 is the empty string. ($*_1$R) can be treated as an additional instance of ($*_2$R) for $n = 0$. (ω) together with ($*_2$R) express *-continuity. In the next section we prove that ACT_ω admits cut-elimination, this means, the set of provable sequents is closed under the rule:

$$\text{(CUT)}\ \frac{\Gamma \Rightarrow A \quad \Gamma_1, A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma, \Gamma_2 \Rightarrow B}.$$

For some finitary fragments of ACT_ω, the cut-elimination theorem has been proved by Lambek [6] and Ono [7].

Let $\mathcal{A}$ be an action lattice. Homomorphisms from the free algebra of formulas to $\mathcal{A}$ are called *assignments* in $\mathcal{A}$. One sets $f(A; B) = f(A) \cdot f(B)$. Assignments are extended to string of formulas by setting:

$$f(\varepsilon) = 1, f(A_1, \ldots, A_n) = f(A_1) \cdot \ldots \cdot f(A_n).$$

A sequent $\Gamma \Rightarrow A$ is said to be *true* in model $(\mathcal{A}, f)$ if $f(\Gamma) \leq f(A)$, and *valid* in $\mathcal{A}$, if it is true in $(\mathcal{A}, f)$ for any assignment f.
By ACT^- we denote the system ACT_ω without the rule (ω). ACT denotes the set of sequents valid in all action lattices. Clearly, $ACT^- \subset ACT \subset ACT_\omega$. We will show that both inclusions are strict.

A sequent $\Gamma \Rightarrow A$ expresses a formula $s \leq t$ of the first-order language of action lattices (s, t are terms), so it expresses an equation $s \vee t = t$. Conversely, any equation $s = t$ is equivalent to $s \leq t$ and $t \leq s$, whence it can be expressed by two sequents. Therefore, ACT_ω is an axiom system for the equational theory of *-continuous action lattices.

In this paper $\vdash_{ACT_\omega} \Gamma \Rightarrow A$ will mean that $\Gamma \Rightarrow A$ is provable in ACT_ω.

3. The cut-elimination theorem

Our purpose is to prove the cut-elimination theorem for ACT_ω.

We define a transfinite chain S_α of sets of sequents. Let S_0 be the set of all axiomatic sequents. We set:

$$S_{\alpha+1} = S_\alpha \cup \{\Gamma \Rightarrow A : \Gamma \Rightarrow A \text{ is the conclusion of some rule whose all premises belong to } S_\alpha\}.$$

For limit ordinals λ we set:

$$S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$$

Then, $S_{\omega_1} = \bigcup_{\alpha<\omega_1} S_\alpha$ is the set of all provable sequents.

The ordinal $r(\Gamma \Rightarrow A) = min\{\alpha : (\Gamma \Rightarrow A) \in S_\alpha\}$ is called *the rank* of a provable sequent $\Gamma \Rightarrow A$.

Theorem 3.1. [the cut-elimination theorem]
The set of sequents provable in ACT_ω is closed under (CUT).

Proof:
We show that the rule (CUT) is admissible in the system ACT_ω, this means

$$\text{if } \vdash_{ACT_\omega} \Psi \Rightarrow C \text{ and } \vdash_{ACT_\omega} \Phi_1, C, \Phi_2 \Rightarrow D, \text{ then } \vdash_{ACT_\omega} \Phi_1, \Psi, \Phi_2 \Rightarrow D \tag{24}$$

We use the following fact: if $\Gamma \Rightarrow A$ is a provable sequent of rank α, then it is an axiom or the conclusion of an inference rule whose every premise is of a rank less than α.
We proceed by a triple induction:
(1) on the complexity of C,
(2) on the rank of the right premise of (CUT): $\Phi_1, C, \Phi_2 \Rightarrow D$,
(3) on the rank of the left premise of (CUT): $\Psi \Rightarrow C$.

We switch on induction (1). We consider some interesting cases. In each case, if either $\Psi \Rightarrow C$, or $\Phi_1, C, \Phi_2 \Rightarrow D$ is an axiom (Id), then the conclusion of (CUT) coincides with one of the premises of (CUT). If $\Psi \Rightarrow C$ is an axiom (0L), then the conclusion of (CUT) is so.

Case 1. $C \equiv A^*$. We switch on induction (2). There is one interesting case: if $\Phi_1, A^*, \Phi_2 \Rightarrow D$ is the conclusion of rule (ω) introducing A^*, then we switch on induction (3). There are three interesting cases: (A) $\Psi \Rightarrow C$ is an axiom ($*_1$R), (B) $\Psi \Rightarrow C$ is the conclusion of ($*_2$R), (C): $\Psi \Rightarrow C$ is the conclusion of (ω). For (A), $\Psi = \varepsilon$, then $\Phi_1, \Psi, \Phi_2 \Rightarrow D$ is one of premises of (ω). For (B), the premises of this rule are $\Psi_1 \Rightarrow A, \ldots, \Psi_n \Rightarrow A$, where $\Psi = \Psi_1 \cdots \Psi_n$. But one of premises of (ω) is $\Phi_1, A^n, \Phi_2 \Rightarrow D$. Applying n times the hypothesis of induction (1) we get $\Phi_1, \Psi_1, \ldots, \Psi_n, \Phi_2 \Rightarrow D$, which is the conclusion of (CUT). For (C), the premises of this rule are $\Psi_1, B^n, \Psi_2 \Rightarrow C$, for any $n \in \omega$, where $\Psi = \Psi_1 B^* \Psi_2$. By induction (3), the sequents $\Phi_1, \Psi_1, B^n, \Psi_2, \Phi_2 \Rightarrow D$ are provable, for any $n \in \omega$. By (ω), $\vdash_{ACT_\omega} \Phi_1, \Psi_1, B^*, \Psi_2, \Phi_2 \Rightarrow D$, which is the conclusion of (CUT).

Case 2. $C \equiv A_1; A_2$. We switch on induction (2). There are three interesting cases: (A) $\Phi_1, C, \Phi_2 \Rightarrow D$ is the conclusion of ($*_2$R), (B) $\Phi_1, C, \Phi_2 \Rightarrow D$ is the conclusion of (ω), (C) $\Phi_1, C, \Phi_2 \Rightarrow D$ is the conclusion of (;L) introducing C. For (A), C has to be in some of the premises of the ($*_2$R)-rule. The premises are $\Gamma_1 \Rightarrow A, \ldots, \Gamma_n \Rightarrow A$, where $\Phi_1 C \Phi_2 = \Gamma_1 \ldots \Gamma_n$ and $D = A^*$. Then, for some i, Γ_i is of the form $\Gamma_i' C \Gamma_i''$. By induction (2), $\vdash_{ACT_\omega} \Gamma_i', \Psi, \Gamma_i'' \Rightarrow A$. Whence, by ($*_2$R), we obtain $\vdash_{ACT_\omega} \Phi_1, \Psi, \Phi_2 \Rightarrow D$. For (B), the premises are of the form $\Gamma_1, A^n, \Gamma_2 \Rightarrow D$ for any $n \in \omega$, where $\Phi_1 C \Phi_2 = \Gamma_1 A^* \Gamma_2$. Then the formula C is in Γ_1 or Γ_2, since $C \neq A^*$. Consider the first case. We have $\Gamma_1 = \Gamma_1' C \Gamma_1''$. We know that each of the sequents $\Gamma_1, A^n, \Gamma_2 \Rightarrow D$ have a rank smaller than $\Phi_1, C, \Phi_2 \Rightarrow D$. By the second induction, $\vdash_{ACT_\omega} \Gamma_1', \Psi, \Gamma_1'', A^n, \Gamma_2 \Rightarrow D$, for every $n \in \omega$. By (ω), we get $\vdash_{ACT_\omega} \Gamma_1', \Psi, \Gamma_1'', A^*, \Gamma_2 \Rightarrow D$, which is the conclusion of (CUT). For (C), the premise is $\Phi_1, A_1, A_2, \Phi_2 \Rightarrow D$. Then, we switch on induction (3). There are two interesting subcases: (C.1) $\Psi \Rightarrow A_1; A_2$ is the conclusion of (;R), (C.2) $\Psi \Rightarrow A_1; A_2$ is the conclusion of (ω). For (C.1), the premises are $\Psi_1 \Rightarrow A_1$, $\Psi_2 \Rightarrow A_2$, where $\Psi = \Psi_1 \Psi_2$. By the first induction, we get $\vdash_{ACT_\omega} \Phi_1, \Psi_1, A_2, \Phi_2 \Rightarrow D$, and then $\vdash_{ACT_\omega} \Phi_1, \Psi_1, \Psi_2, \Phi_2 \Rightarrow D$, which is the conclusion of (CUT). For (C.2), the premises are $\Psi_1, A^n, \Psi_2 \Rightarrow C$, for any $n \in \omega$, where $\Psi = \Psi_1 A^* \Psi_2$. By induction (3), the sequents $\Phi_1, \Psi_1, A^n, \Psi_2, \Phi_2 \Rightarrow D$ are provable in ACT_ω, for any $n \in \omega$. By (ω), we have $\vdash_{ACT_\omega} \Phi_1, \Psi_1, A^*, \Psi_2, \Phi_2 \Rightarrow D$, which is the conclusion of (CUT).

Case 3. $C \equiv A_2 \leftarrow A_1$. We switch on induction (2). There is one interesting case: $\Phi_1, C, \Phi_2 \Rightarrow D$ is the conclusion of the ($\leftarrow$L)-rule introducing C. The premises are $\Phi' \Rightarrow A_1$ and $\Phi_1, A_2, \Phi'' \Rightarrow D$, where $\Phi_2 = \Phi'\Phi''$. Like in case 2 we switch on induction (3). There is one interesting case when $\Psi \Rightarrow A_2 \leftarrow A_1$ is the conclusion of the ($\leftarrow$R)-rule with the premise $\Psi, A_1 \Rightarrow A_2$. By the hypothesis of induction (1), we get $\vdash_{ACT_\omega} \Phi_1, \Psi, A_1, \Phi'' \Rightarrow D$ and $\vdash_{ACT_\omega} \Phi_1, \Psi, \Phi', \Phi'' \Rightarrow D$, which is the conclusion of (CUT).

Case 4. $C \equiv 0$. We switch on induction (2). There is one interesting case: $\Phi_1, C, \Phi_2 \Rightarrow D$ is an axiom (0L). Like in case 2 we switch on induction (3). There is one interestiong subcase when $\Psi \Rightarrow 0$ is the conclusion of another rule. We directly apply the hypothesis of induction (3).

Case 5. $C \equiv p$. We switch on induction (2). There are two interesting cases: (A) $\Phi_1, p, \Phi_2 \Rightarrow D$ is an axiom (0L), (B) $\Phi_1, p, \Phi_2 \Rightarrow D$ is the conclusion of some rule. For (A), 0 is in Φ_1 or Φ_2. Consider the first case. We have $\Phi_1 = \Phi' 0 \Phi''$, so $\Phi_1, \Psi, \Phi_2 \Rightarrow D$ is an axiom (0L). For (B), p appears in some premise of this rule (in each premise of (ω), ($\vee$L), ($\wedge$R)), whence we directly apply the hypothesis of induction (2) and this rule. □

4. Infinitary action logic

This section studies some properties of ACT_ω. We introduce some definitions and lemmas helpful for proving main theorems of this paper.

Lemma 4.1.

1. $\vdash_{ACT_\omega} \Phi_1, A; B, \Phi_2 \Rightarrow C$ iff $\vdash_{ACT_\omega} \Phi_1, A, B, \Phi_2 \Rightarrow C$
2. $\vdash_{ACT_\omega} \Phi_1, A \vee B, \Phi_2 \Rightarrow C$ iff $\vdash_{ACT_\omega} \Phi_1, A, \Phi_2 \Rightarrow C$ and $\vdash_{ACT_\omega} \Phi_1, B, \Phi_2 \Rightarrow C$

Proof:
The first part holds by (;R), (;L) and (CUT). The second part holds by ($\vee_1$R), ($\vee_2$R), ($\vee$L) and (CUT). □

The following rules are derivable in ACT_ω:

$$\frac{A \Rightarrow B \quad C \Rightarrow D}{A; C \Rightarrow B; D} \tag{25}$$

$$\frac{A \Rightarrow B \quad C \Rightarrow D}{B \rightarrow C \Rightarrow A \rightarrow D} \tag{26}$$

$$\frac{A \Rightarrow B \quad C \Rightarrow D}{C \leftarrow B \Rightarrow D \leftarrow A} \tag{27}$$

$$\frac{A \Rightarrow B \quad C \Rightarrow D}{A \vee C \Rightarrow B \vee D} \tag{28}$$

$$\frac{A \Rightarrow B \quad C \Rightarrow D}{A \wedge C \Rightarrow B \wedge D} \tag{29}$$

$$\frac{A \Rightarrow B}{A^* \Rightarrow B^*} \tag{30}$$

Let us show two of them. We consider (30). By ($*_1$R) and ($*_2$R), for any $n \in \omega$, we get

$$\frac{A \Rightarrow B, \ldots, A \Rightarrow B}{A^n \Rightarrow B^*}$$

Then, by the rule (ω), $A^* \Rightarrow B^*$. Now let us consider the rule (28). From ($\vee_2$R) and ($\vee_1$R) we have

$$\frac{A \Rightarrow B}{A \Rightarrow B \vee D} \text{ and } \frac{C \Rightarrow D}{C \Rightarrow B \vee D}.$$

Then, $A \vee C \Rightarrow B \vee D$, by ($\vee$L).

We define an equivalence relation $\sim$ on the algebra of formulas as follows:

$$A \sim B \text{ iff } \vdash_{ACT_\omega} A \Rightarrow B \text{ and } \vdash_{ACT_\omega} B \Rightarrow A$$

where A, B are formulas. By (25)-(30), $\sim$ is *a congruence* in this algebra. We construct a quotient structure $M = FOR/\sim$. We set:

$$\begin{gathered}
[A] = \{B : A \sim B\} \\
M = \{[A] : A \in FOR\} \\
[A] \vee [B] = [A \vee B] \\
[A] \wedge [B] = [A \wedge B] \\
[A] \cdot [B] = [A; B] \\
[A] \to [B] = [A \to B] \\
[A] \leftarrow [B] = [A \leftarrow B] \\
([A])^* = [A^*] \\
\mathbf{1} = [1] \\
\mathbf{0} = [0]
\end{gathered}$$

We consider the algebra $\mathcal{M} = (M, \vee, \wedge, \cdot, ()^*, \to, \leftarrow, 1, 0)$. It is easy to show that $\mathcal{M}$ is an action lattice. We define an assignment $\mu : FOR \to M$ such that $\mu(A) = [A]$ for $A \in FOR$.

Theorem 4.1. The sequents provable in ACT_ω are precisely the sequents true in all *-continuous action lattices.

Proof:
($\Rightarrow$) If $\vdash_{ACT_\omega} \Gamma \Rightarrow A$, then $\Gamma \Rightarrow A$ is true in all *-continuous action lattices. It is easy to see that axioms and all rules except (ω) are true in all action lattices. The rule (ω) is true in any *-continuous action lattice, because we have the following condition:

$$\text{if } (xa^n y \leq z, \text{ for any } n \in \omega), \text{ then } xa^* y \leq z$$

($\Leftarrow$) In the quotient structure $\mathcal{M}$, defined above, we have $[A] \leq [B]$ iff $[A] \vee [B] = [B]$ iff $[A \vee B] = [B]$ iff $\vdash_{ACT_\omega} A \vee B \Rightarrow B$ iff $\vdash_{ACT_\omega} A \Rightarrow B$. If $\nvdash_{ACT_\omega} A \Rightarrow B$, then $[A] \not\leq [B]$, so the sequent $A \Rightarrow B$ is not true in model $(\mathcal{M}, \mu)$. By (ω), $\mathcal{M}$ is a *-continuous. □

Formulas of the form B^* are called *-formulas. We define $A^{\leq n} \equiv A^0 \vee \cdots \vee A^n$, for $n \in \omega$. (We set $A^0 \equiv 1$ and, for $n \geq 1$, A^n is the product of n copies of A.) Let $P_n(A)$ (resp. $N_n(A)$) be the formula arising from A by a (successive) replacement of any positive (resp. negative) *-subformula B^* by $B^{\leq n}$. This rough formulation can be replaced by a strict, recursive definition given as follows

$$\begin{gathered}
P_n(p) \equiv p,\, N_n(p) \equiv p \\
P_n(0) \equiv 0,\, N_n(0) \equiv 0 \\
P_n(1) \equiv 1,\, N_n(1) \equiv 1 \\
P_n(A_1 \to A_2) \equiv N_n(A_1) \to P_n(A_2) \\
N_n(A_1 \to A_2) \equiv P_n(A_1) \to N_n(A_2) \\
P_n(A_2 \leftarrow A_1) \equiv P_n(A_2) \leftarrow N_n(A_1) \\
N_n(A_2 \leftarrow A_1) \equiv N_n(A_2) \leftarrow P_n(A_1) \\
P_n(A_1 \wedge A_2) \equiv P_n(A_1) \wedge P_n(A_2) \\
N_n(A_1 \wedge A_2) \equiv N_n(A_1) \wedge N_n(A_2) \\
P_n(A_1 \vee A_2) \equiv P_n(A_1) \vee P_n(A_2) \\
N_n(A_1 \vee A_2) \equiv N_n(A_1) \vee N_n(A_2) \\
P_n(A_1; A_2) \equiv P_n(A_1); P_n(A_2) \\
N_n(A_1; A_2) \equiv N_n(A_1); N_n(A_2) \\
P_n(C^*) \equiv (P_n(C))^{\leq n} \\
N_n(C^*) \equiv (N_n(C))^*
\end{gathered}$$

We set

$$\begin{gathered}
P_n(A_1, \ldots, A_k) = P_n(A_1), \ldots, P_n(A_k) \\
N_n(\Gamma \Rightarrow A) = P_n(\Gamma) \Rightarrow N_n(A)
\end{gathered}$$

Lemma 4.2. In ACT_ω we can restrict axioms (Id) to sequents $p \Rightarrow p$ such that p is a variable.

Proof:
We show that $A \Rightarrow A$ is provable. The proof is by induction on the complexity of A. We consider some cases.

Case 1. $A \equiv 1$. We have

$$\frac{\Rightarrow 1}{1 \Rightarrow 1}(1L).$$

Case 2. $A \equiv 0$. We have $0 \Rightarrow 0$, by (0L).

Case 3. $A \equiv p$. It's obvious.

Case 4. $A \equiv A_1 \vee A_2$, $A_1 \wedge A_2$, $A_1; A_2$, $A_1 \to A_2$, $A_2 \leftarrow A_1$, B^*. We use the induction hypothesis and rules (25)-(30). □

Lemma 4.3. $\vdash_{ACT_\omega} P_n(A) \Rightarrow A, \vdash_{ACT_\omega} A \Rightarrow N_n(A)$

Proof:
We have $\vdash_{ACT_\omega} B^n \Rightarrow B^*$. By (;L), ($\vee$L), (1L) and ($*_1$R), we get $\vdash_{ACT_\omega} B^{\leq n} \Rightarrow B^*$. The proof goes by induction on the complexity of A, using (25)-(30). □

Fact 4.1. If $m \leq n$, then $\vdash_{ACT_\omega} A^{\leq m} \Rightarrow A^{\leq n}$.

Lemma 4.4. If $m \leq n$ then $\vdash_{ACT_\omega} P_m(A) \Rightarrow P_n(A)$ and $\vdash_{ACT_\omega} N_n(A) \Rightarrow N_m(A)$.

Proof:
It is similar to the proof of lemma 4.3. □

Lemma 4.5. If $m \leq n$ and $\vdash_{ACT_\omega} N_n(\Gamma \Rightarrow A)$, then $\vdash_{ACT_\omega} N_m(\Gamma \Rightarrow A)$.

Proof:
Assume $m \leq n$ and $\vdash_{ACT\omega} N_n(\Gamma \Rightarrow A)$. Let $\Gamma = (A_1, \ldots, A_k)$. By lemma 4.4 $\vdash_{ACT_\omega} P_m(\Gamma) \Rightarrow P_n(\Gamma)$, which means that $\vdash_{ACT_\omega} P_m(A_i) \Rightarrow P_n(A_i)$ for any $i = 1, \ldots, k$. By (CUT) and lemma 4.4, we get $P_m(\Gamma) \Rightarrow N_m(A)$, so $\vdash_{ACT_\omega} N_m(\Gamma \Rightarrow A)$. □

The proofs of lemmas 4.6, 4.7 and 4.8 are straightforward, by the definition of P_n and N_n.

Lemma 4.6.

1. Let $\circ \in \{\wedge, ;\}$. If $P_n(A) \equiv B_1 \circ B_2$, then there exist A_1, A_2 such that $A \equiv A_1 \circ A_2$ and $P_n(A_i) \equiv B_i$, for $i = 1, 2$.
2. If A is not a *-formula and $P_n(A) \equiv B_1 \vee B_2$, then there exist A_1, A_2 such that $A \equiv A_1 \vee A_2$ and $P_n(A_i) \equiv B_i$, for $i = 1, 2$.
3. If $P_n(A) \equiv B_1 \rightarrow B_2$ (resp. $P_n(A) \equiv B_2 \leftarrow B_1$), then there exist A_1, A_2 such that $A \equiv A_1 \rightarrow A_2$ (resp. $A \equiv A_2 \leftarrow A_1$) and $N_n(A_1) \equiv B_1$, $P_n(A_2) \equiv B_2$.

Lemma 4.7.

1. Let $\circ \in \{\wedge, \vee, ;\}$. If $N_n(A) \equiv B_1 \circ B_2$, then there exist A_1, A_2 such that $A \equiv A_1 \circ A_2$ and $N_n(A_i) \equiv B_i$, for $i = 1, 2$.
2. If $N_n(A) \equiv B^*$, then there exists C such that $A \equiv C^*$ and $N_n(C) \equiv B$.
3. If $N_n(A) \equiv B_1 \rightarrow B_2$ (resp. $N_n(A) \equiv B_2 \leftarrow B_1$), then there exist A_1, A_2 such that $A \equiv A_1 \rightarrow A_2$ (resp. $A \equiv A_2 \leftarrow A_1$) and $P_n(A_1) \equiv B_1$, $N_n(A_2) \equiv B_2$.

Lemma 4.8.

1. If A is not a *-formula and $P_n(A) \equiv 1$, then $A \equiv 1$.
2. If $P_n(A) \equiv p$ (resp. $P_n(A) \equiv 0$), then $A \equiv p$ (resp. $A \equiv 0$).
3. If $N_n(A) \equiv p$ (resp. $N_n(A) \equiv 0$, $N_n(A) \equiv 1$), then $A \equiv p$ (resp. $A \equiv 0$, $A \equiv 1$).

All rules of ACT_ω except the rule (ω) are said to be *finitary rules*.
Let X be a set of sequents. We set

$$N_n(X) = \{N_n(\Gamma \Rightarrow A) : (\Gamma \Rightarrow A) \in X\}$$

The following lemma is crucial.

Lemma 4.9. Let $N_n(\Phi \Rightarrow A)$ be the conclusion of a finitary rule R with the set of premises X, and assume that no formula in string Φ is a *-formula. Then, $\Phi \Rightarrow A$ is the conclusion of rule R with some set of premises Y such that $X = N_n(Y)$.

Proof:
One must examine all finitary rules. We consider seven cases. Remind that $N_n(\Phi \Rightarrow A) = P_n(\Phi) \Rightarrow N_n(A)$.

Case 1. Rule ($\wedge$R). The inference looks as follows:

$$\frac{P_n(\Phi) \Rightarrow B_1 \quad P_n(\Phi) \Rightarrow B_2}{P_n(\Phi) \Rightarrow B_1 \wedge B_2}$$

where $N_n(A) \equiv B_1 \wedge B_2$. By lemma 4.7.1, there exist A_1, A_2 such that $A \equiv A_1 \wedge A_2$ and $N_n(A_i) \equiv B_i$, for $i = 1, 2$. The inference:

$$\frac{\Phi \Rightarrow A_1 \quad \Phi \Rightarrow A_2}{\Phi \Rightarrow A_1 \wedge A_2}$$

fits the scheme of ($\wedge$R). For $X = \{P_n(\Phi) \Rightarrow B_1, P_n(\Phi) \Rightarrow B_2\}$, $Y = \{\Phi \Rightarrow A_1, \Phi \Rightarrow A_2\}$, we get $X = N_n(Y)$.

Case 2. Rule (;R). The inference looks as follows:

$$\frac{P_n(\Phi_1) \Rightarrow B_1 \quad P_n(\Phi_2) \Rightarrow B_2}{P_n(\Phi_1), P_n(\Phi_2) \Rightarrow B_1; B_2}$$

where $\Phi = \Phi_1\Phi_2$ and $N_n(A) \equiv B_1; B_2$. By lemma 4.7.1, there exist A_1, A_2 such that $A \equiv A_1; A_2$ and $N_n(A_i) \equiv B_i$, for $i = 1, 2$. The inference:

$$\frac{\Phi_1 \Rightarrow A_1 \quad \Phi_2 \Rightarrow A_2}{\Phi \Rightarrow A_1; A_2}$$

fits the scheme of (;R). For $X = \{P_n(\Phi_1) \Rightarrow B_1, P_n(\Phi_2) \Rightarrow B_2\}$, $Y = \{\Phi_1 \Rightarrow A_1, \Phi_2 \Rightarrow A_2\}$, we get $X = N_n(Y)$.

Case 3. Rule (;L). The inference looks as follows:

$$\frac{\Gamma_1, B, C, \Gamma_2 \Rightarrow N_n(A)}{\Gamma_1, B; C, \Gamma_2 \Rightarrow N_n(A)}$$

Hence $\Phi = \Phi_1 D \Phi_2$, where $P_n(\Phi_1) \equiv \Gamma_1$, $P_n(\Phi_2) \equiv \Gamma_2$ and $P_n(D) \equiv B; C$. By lemma 4.6.1, there exist $B^{'}, C^{'}$ such that $D \equiv B^{'}; C^{'}$ and $P_n(B^{'}) \equiv B$, $P_n(C^{'}) \equiv C$. The inference:

$$\frac{\Phi_1, B^{'}, C^{'}, \Phi_2 \Rightarrow A}{\Phi_1, B'; C', \Phi_2 \Rightarrow A}$$

fits the scheme of (;L). For $X = \{\Gamma_1, B, C, \Gamma_2 \Rightarrow N_n(A)\}$, $Y = \{\Phi_1, B^{'}, C^{'}, \Phi_2 \Rightarrow A\}$, we get $X = N_n(Y)$.

Case 4. Rule (→R). The inference looks as follows:

$$\frac{B_1, P_n(\Phi) \Rightarrow B_2}{P_n(\Phi) \Rightarrow B_1 \to B_2}$$

where $N_n(A) \equiv B_1 \to B_2$. By lemma 4.7.3, there exist A_1, A_2 such that $A \equiv A_1 \to A_2$ and $P_n(A_1) \equiv B_1$, $N_n(A_2) \equiv B_2$. The inference:

$$\frac{A_1, \Phi \Rightarrow A_2}{\Phi \Rightarrow A_1 \to A_2}$$

fits the scheme of (→R). For $X = \{B_1, P_n(\Phi) \Rightarrow B_2\}$, $Y = \{A_1, \Phi \Rightarrow A_2\}$, we get $X = N_n(Y)$.

Case 5. Rule ($*_2$R). The inference looks as follows:

$$\frac{P_n(\Phi_1) \Rightarrow B, \ldots, P_n(\Phi_k) \Rightarrow B}{P_n(\Phi_1), \ldots, P_n(\Phi_k) \Rightarrow B^*}$$

where $\Phi = \Phi_1 \ldots \Phi_k$ and $N_n(A) \equiv B^*$. By lemma 4.7.2, there exists C such that $A \equiv C^*$ and $N_n(C) \equiv B$. The inference:

$$\frac{\Phi_1 \Rightarrow C, \ldots, \Phi_k \Rightarrow C}{\Phi \Rightarrow C^*}$$

fits the scheme of ($*_2$R). For $X = \{P_n(\Phi_1) \Rightarrow B, \ldots, P_n(\Phi_k) \Rightarrow B\}$, $Y = \{\Phi_1 \Rightarrow C, \ldots, \Phi_k \Rightarrow C\}$, we get $X = N_n(Y)$.

Case 6. Rule (∨L). The inference looks as follows:

$$\frac{\Gamma_1, B, \Gamma_2 \Rightarrow N_n(A) \quad \Gamma_1, C, \Gamma_2 \Rightarrow N_n(A)}{\Gamma_1, B \vee C, \Gamma_2 \Rightarrow N_n(A)}$$

Hence $\Phi = \Phi_1 D \Phi_2$, where $P_n(\Phi_1) \equiv \Gamma_1$, $P_n(\Phi_2) \equiv \Gamma_2$ and $P_n(D) \equiv B \vee C$. By lemma 4.6.2, there exist $B^{'}, C^{'}$ such that $D \equiv B^{'} \vee C^{'}$ and $P_n(B^{'}) \equiv B$, $P_n(C^{'}) \equiv C$. The inference:

$$\frac{\Phi_1, B^{'}, \Phi_2 \Rightarrow A \quad \Phi_1, C^{'}, \Phi_2 \Rightarrow A}{\Phi_1, B' \vee C', \Phi_2 \Rightarrow A}$$

fits the scheme of (∨L). For $X = \{\Gamma_1, B, \Gamma_2 \Rightarrow N_n(A),\ \Gamma_1, C, \Gamma_2 \Rightarrow N_n(A)\}$, $Y = \{\Phi_1, B^{'}, \Phi_2 \Rightarrow A,\ \Phi_1, C^{'}, \Phi_2 \Rightarrow A\}$, we get $X = N_n(Y)$.

Case 7. Rule (1L). The inference looks as follows:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow N_n(A)}{\Gamma_1, 1, \Gamma_2 \Rightarrow N_n(A)}$$

Hence $\Phi = \Phi_1 D \Phi_2$, where $P_n(\Phi_1) \equiv \Gamma_1$, $P_n(\Phi_2) \equiv \Gamma_2$ and $P_n(D) \equiv 1$. By lemma 4.8.1, $D \equiv 1$. The inference:

$$\frac{\Phi_1, \Phi_2 \Rightarrow A}{\Phi_1, 1, \Phi_2 \Rightarrow A}$$

fits the scheme of (1L). For $X = \{\Gamma_1, \Gamma_2 \Rightarrow N_n(A)\}$, $Y = \{\Phi_1, \Phi_2 \Rightarrow A\}$, we get $X = N_n(Y)$. □

5. The *-elimination theorem

Our purpose is to prove the theorem on elimination of negative occurrences of * in ACT_ω.

Let us introduce some auxiliary notions. The complexity of a formula is the total number of occurrences of symbols $\vee, \wedge, ;, \rightarrow, \leftarrow, ^*, 0, 1$ in this formula.

To each sequent $\Gamma \Rightarrow A$ we assign a string of integers $c(\Gamma \Rightarrow A) \in \omega^*$ such that $c(\Gamma \Rightarrow A) = (c_0, c_1, \ldots, c_r)$, $c_i \in \omega$ and $c_r \neq 0$, where c_i is the number of all occurrences of formulas of complexity i in $\Gamma \Rightarrow A$, and r is the biggest complexity of a formula in the sequent. $c(\Gamma \Rightarrow A)$ is called *the complexity* of $\Gamma \Rightarrow A$.

We define an ordering relation on ω^*

1. if $r < s$, then $(c_0, \ldots, c_r) < (d_0, \ldots, d_s)$
2. if $r = s$, then $(c_0, \ldots, c_r) < (d_0, \ldots, d_r)$ iff $c_{max\{i:c_i \neq d_i\}} < d_{max\{i:c_i \neq d_i\}}$ (strings of the same length are arranged in the antilexicographical order).

Thus, $(\omega^*, \leq)$ is a well-ordered set of type ω^ω, where $\omega^\omega = sup\{\omega^n : n \in \omega\}$.

Lemma 5.1. If R is any instance of a rule of ACT_ω, then the complexity of the conclusion of R is bigger than the complexity of each premise of R.

Theorem 5.1. [the *-elimination theorem]
$\vdash_{ACT_\omega} \Gamma \Rightarrow A$ iff, for all $n \in \omega$, $\vdash_{ACT_\omega} N_n(\Gamma \Rightarrow A)$.

Proof:
($\Rightarrow$) holds by lemma 4.3 and (CUT). For ($\Leftarrow$), we show:

$$\text{(!) if } \nvdash_{ACT_\omega} \Gamma \Rightarrow A, \text{ then there exists } n \in \omega \text{ such that } \nvdash_{ACT_\omega} N_n(\Gamma \Rightarrow A)$$

We prove (!) by transfinite induction on $c(\Gamma \Rightarrow A)$. Assume that (!) is true for all sequents $\Gamma' \Rightarrow A'$ such that $c(\Gamma' \Rightarrow A') < c(\Gamma \Rightarrow A)$. We prove (!) for $\Gamma \Rightarrow A$. Assume $\nvdash_{ACT_\omega} \Gamma \Rightarrow A$. Then, $\Gamma \Rightarrow A$ is not an axiom. We consider two cases.

Case 1. $\Gamma = \Gamma_1, B^*, \Gamma_2$. By ($\omega$), there exist $m \in \omega$ such that $\nvdash_{ACT_\omega} \Gamma_1, B^m, \Gamma_2 \Rightarrow A$. Since $c(\Gamma_1, B^m, \Gamma_2 \Rightarrow A) < c(\Gamma \Rightarrow A)$, then, by the induction hypothesis, there exists $k \in \omega$ such that $\nvdash_{ACT_\omega} N_k(\Gamma_1, B^m, \Gamma_2 \Rightarrow A)$. Denote $n = max(m, k)$. By lemma 4.5, $\nvdash_{ACT_\omega} N_n(\Gamma_1, B^m, \Gamma_2 \Rightarrow A)$. Since $\nvdash_{ACT_\omega} P_n(\Gamma_1), (P_n(B))^m, P_n(\Gamma_2) \Rightarrow N_n(A)$, then, by lemma 4.1, we have $\nvdash_{ACT_\omega} P_n(\Gamma_1), (P_n(B))^{\leq n}, P_n(\Gamma_2) \Rightarrow N_n(A)$. Hence $\nvdash_{ACT_\omega} N_n(\Gamma_1, B^*, \Gamma_2 \Rightarrow A)$.

Case 2. No formula in Γ is a *-formula. Let $R_1, \ldots, R_k$ be all instances of rules whose conclusion is $\Gamma \Rightarrow A$, and let Y_i be the set of premises of R_i. From the assumption of case 2 it follows that all rules $R_1, \ldots, R_k$ are finitary and all sets Y_i are finite and nonempty. Since $\nvdash_{ACT_\omega} \Gamma \Rightarrow A$, then in each set Y_i there exists a sequent $(\Phi_i \Rightarrow A_i) \in Y_i$ such that $\nvdash_{ACT_\omega} \Phi_i \Rightarrow A_i$. Since, by lemma 5.1, $c(\Phi_i \Rightarrow A_i) < c(\Gamma \Rightarrow A)$, then, by the induction hypothesis, there exists $n_i \in \omega$ such that $\nvdash_{ACT_\omega} N_{n_i}(\Phi_i \Rightarrow A_i)$. Denote $n = max\{n_i : 1 \leq i \leq k\}$. We show that $\nvdash_{ACT_\omega} N_n(\Gamma \Rightarrow A)$. Suppose that $\vdash_{ACT_\omega} N_n(\Gamma \Rightarrow A)$. We consider two cases.

Case 2.1. $N_n(\Gamma \Rightarrow A)$ is an axiom of ACT_ω. We consider four subcases.
(A) $N_n(\Gamma \Rightarrow A)$ is an axiom (Id): $p \Rightarrow p$. Hence $P_n(\Gamma) = p$ and $N_n(A) = p$. By lemma 4.8, $\Gamma = p$, $A = p$, so $\Gamma \Rightarrow A$ is (Id), which is impossible.
(B) $N_n(\Gamma \Rightarrow A)$ is the axiom (1R): $\Rightarrow 1$. Hence $P_n(\Gamma) = \varepsilon$, $N_n(A) = 1$. So, $\Gamma = \varepsilon$ and, by lemma 4.8, $A = 1$, so $\Gamma \Rightarrow A$ is (1R), which is impossible.
(C) $N_n(\Gamma \Rightarrow A)$ is an axiom ($*_1$R): $\Rightarrow B^*$. Hence $P_n(\Gamma) = \varepsilon$ and $N_n(A) = B^*$. So, $\Gamma = \varepsilon$ and, by lemma 4.7, there exists C such that $A = C^*$ and $N_n(C) = B$. Thus, $\Gamma \Rightarrow A$ is ($*_1$R), which is impossible.
(D) $N_n(\Gamma \Rightarrow A)$ is an axiom (0L): $\Phi_1, 0, \Phi_2 \Rightarrow B$. Hence $\Gamma = \Gamma_1 C \Gamma_2$, where $\Phi_i = P_n(\Gamma_i)$, $i = 1, 2$, $0 = P_n(C)$ and $N_n(A) = B$. By lemma 4.8.2, $C \equiv 0$, whence $\Gamma \Rightarrow A$ is an axiom (0L), which is impossible.

Case 2.2. $N_n(\Gamma \Rightarrow A)$ is the conclusion of a rule R with a set of premises X such that all sequents in X are provable in ACT_ω. Since $N_n(\Gamma \Rightarrow A)$ contains no negative occurrences of *, then R must be a finitary rule. By lemma 4.9, $\Gamma \Rightarrow A$ is the conclusion of the same rule with a set of the premises Y such that $X = N_n(Y)$. This instance of R is on the list $R_1, \ldots, R_k$, so $Y = Y_i$. It follows that there exists $(\Phi_i \Rightarrow A_i) \in Y_i$ such that $\not\vdash_{ACT_\omega} N_n(\Phi_i \Rightarrow A_i)$. Since $N_n(\Phi_i \Rightarrow A_i) \in X$, we get the contradiction. □

Corollary 5.1. The decision problem for ACT_ω is Π^0_1.

Proof:
The relation R is Π^0_1 iff there exists a recursive relation S such that

$$R(x) \Leftrightarrow \text{for all } n \in \omega\ S(n, x)$$

The relation $S(n, \Gamma \Rightarrow A) \Leftrightarrow \vdash_{ACT_\omega} N_n(\Gamma \Rightarrow A)$ is recursive. For $N_n(\Gamma \Rightarrow A)$ has no negative occurrences of *, so it is provable in ACT_ω iff it is provable in ACT^- (see section 2). Then, by theorem 5.1, the relation $\vdash_{ACT_\omega} \Gamma \Rightarrow A$ is Π^0_1. □

Buszkowski [1] reduces the total language problem for context-free grammars to the decision problem for ACT_ω, which yields the Π^0_1-hardness of ACT_ω.

Corollary 5.2. The decision problem for ACT_ω is Π^0_1-complete.

ACT is recursively enumerable, so the inclusion $ACT \subset ACT_\omega$ is strict. Let p be a variable. $p, p^* \Rightarrow p^*$ belongs to ACT, but $p, p^* \Rightarrow p^*$ is not provable in ACT^-. So, the inclusion $ACT^- \subset ACT$ is strict.

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